VIII. *Quaternions and Projective Geometry.*

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Communicated by Sir ROBERT BALL, *F.R.S.*

Received November 27,—Read December 11, 1902.

INTRODUCTION.

A QUATERNION q adequately represents a point Q to which a determinate weight is attributed, and, conversely, when the point and its weight are given, the quaternion is defined without ambiguity. This is evident from the identity

$$q = \left(1 + \frac{Vq}{Sq}\right) Sq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (A),$$

in which Sq is regarded as a weight placed at the extremity of the vector

$$\text{OQ} = \frac{Vq}{Sq} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B),$$

drawn from any assumed origin O. It is sometimes convenient to employ capitals Q concurrently with italics q to represent the same point, it being understood that

$$\text{Q} = \frac{q}{Sq} = 1 + \text{OQ} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (C).$$

Thus Q represents the point Q affected with a unit weight. The point O may be called the *scalar* point, for we have

$$\text{O} = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (D).$$

In order to develop the method, it becomes necessary to employ certain special symbols. With one exception these are found in Art. 365 of 'Hamilton's Elements of Quaternions,' though in quite a different connection. We write

$$(a, b) = bSa - aSb, \quad [a, b] = V \,.\, VaVb \quad . \quad . \quad . \quad . \quad . \quad . \quad (E);$$

and in particular for points of unit weight, these become

$$(\text{A}, \text{B}) = \text{B} - \text{A}, \quad [\text{A}, \text{B}] = V.V\text{A}V\text{B} = V.V\text{A}\,.\,(\text{B} - \text{A}) \quad . \quad . \quad . \quad (F).$$

Thus (ab) is the product of the weights $SaSb$ into the vector connecting the points, and $[ab]$ is the product of the weights into the moment of the vector connecting the points with respect to the scalar point. The two functions (ab) and $[ab]$ completely define the line ab.

Again HAMILTON writes

$$[a,b,c]=(a,b,c)-[b,c]\mathrm{S}a-[c,a]\mathrm{S}b-[a,b]\mathrm{S}c;\ (a,b,c)=\mathrm{S}[a,b,c]=\mathrm{SV}a\mathrm{V}b\mathrm{V}c\,. \quad \text{(G)};$$

or if we replace a, b, c by $(1+\alpha)\mathrm{S}a$, $(1+\beta)\mathrm{S}b$, $(1+\gamma)\mathrm{S}c$, where α, β and γ are the vectors from the scalar point to three points a, b and c, we have

$$[\mathrm{A}, \mathrm{B}, \mathrm{C}] = \mathrm{S}\alpha\beta\gamma - \mathrm{V}(\beta\gamma + \gamma\alpha + \alpha\beta);\ (\mathrm{A}, \mathrm{B}, \mathrm{C}) = \mathrm{S}\alpha\beta\gamma \quad . \quad . \quad . \quad \text{(H)}.$$

Hence it appears that $[a, b, c]$ is the symbol of the plane a, b, c; for $-\mathrm{V}[a, b, c](a, b, c)^{-1}$ is the reciprocal of the vector perpendicular from the scalar point on that plane. Also $(\mathrm{A}, \mathrm{B}, \mathrm{C})$ is the sextupled volume of the tetrahedron OABC.

Again, HAMILTON writes for four quaternions

$$(abcd) = \mathrm{S}\,.\,a[bcd] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(I)};$$

and in terms of the vectors this is seen to be the products of the weights into the sextupled volume of the pyramid (ABCD).

Other notations may of course be employed for these five combinatorial functions of two, three, or four quaternions or points, but HAMILTON'S use of the brackets seems to be quite satisfactory.

In the same article HAMILTON gives two most useful identities connecting any five quaternions. These are

$$a(bcde) + b(cdea) + c(deab) + d(eabc) + e(abcd) = 0\,. \quad . \quad . \quad . \quad \text{(J)},$$

and

$$e(abcd) = [bcd]\mathrm{S}ae - [acd]\mathrm{S}be + [abd]\mathrm{S}ce - [abc]\mathrm{S}de \quad . \quad . \quad . \quad \text{(K)},$$

which enable us to express any point in terms of any four given points, or in terms of any four given planes.

The equation of a plane may be written in the form

$$\mathrm{S}lq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(L)};$$

and thus l, any quaternion whatever, may be regarded as the symbol of a plane as well as of a point.

On the whole, it seems most convenient to take as the auxiliary quadric the sphere of unit radius

$$\mathrm{S}\,.\,q^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(M)},$$

whose centre is the scalar point. With this convention the plane $\mathrm{S}lq = 0$ is the polar of the point l with respect to the auxiliary quadric; or the plane is the reciprocal of the point l. Thus the principle of duality occupies a prominent position.

The formulæ of reciprocation

$$([abc];[abd]) = [ab](abcd);\ [[abc];[abd]] = -(ab)(abcd)\,. \quad . \quad . \quad \text{(N)}$$

connecting any four quaternions are worthy of notice, and are easily proved by

replacing the quaternions by $1 + \alpha$, $1 + \beta$, $1 + \gamma$, and $1 + \delta$ respectively. In complicated relations it may be safer to separate the quaternions as in these formulæ by semi-colons, but generally the commas or semi-colons may be omitted without causing any ambiguity.

These new interpretations are not in the least inconsistent with any principle of the calculus of quaternions. We are still at liberty to regard a quaternion as the separable sum of a vector and a scalar, or as the ratio or product of two vectors, or as an operator, as well as a symbol of a point or of a plane.

In particular, in addition to HAMILTON'S definition of a vector as a right line of given direction and of given magnitude, and in addition to his subsequent interpretations of a vector as the ratio or product of two mutually rectangular vectors, or as a versor, we may now consider a vector as denoting the point at infinity in its direction, or the plane through the centre of reciprocation. For the vector OQ of equation (B) becomes infinitely long if $Sq = 0$, and the plane $Slq = 0$ passes through the scalar point if $Sl = 0$. We may also observe that the difference of two unit points A — B is the vector from one point B to the other A, and this again is in agreement with the opening sections of the "Lectures."

Additional illustrations and examples may be found in a paper on "The Interpretation of a Quaternion as a Point-symbol," 'Trans. Roy. Irish Acad.,' vol. 32, pp. 1–16.

The only other symbols peculiar to this method are the symbols for quaternion arrays. The five functions (ab), $[ab]$, $[abc]$, (abc), and $(abcd)$ are particular cases of arrays, being, in fact, arrays of one row. In general the array of m rows and n columns

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & \ldots & a_n \\ b_1 & b_2 & b_3 & \ldots & b_n \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ \cdot & \cdot & \cdot & \ldots & \cdot \\ p_1 & p_2 & p_3 & \ldots & p_n \end{matrix}\right\} \quad \ldots \ldots \ldots \quad (O)$$

may be defined as a function of mn quaternion constituents, which vanishes if, and only if, the groups of the constituents composing the rows were connected by linear relations with the same set of scalar multipliers. In other words, the array vanishes if scalars $t_1, t_2 \ldots t_n$ can be found to satisfy the m equations

$$\begin{aligned} t_1 a_1 + t_2 a_2 + \ldots + t_n a_n &= 0, \\ t_1 b_1 + t_2 b_2 + \ldots + t_n b_n &= 0, \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \ldots & \quad \cdot \quad \cdot \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \ldots & \quad \cdot \quad \cdot \\ t_1 p_1 + t_2 p_2 + \ldots + t_n p_n &= 0. \end{aligned}$$

The expansion of arrays is considered in a paper on "Quaternion Arrays," 'Trans. Roy. Irish Acad.,' vol. 32, pp. 17–30.

SECTION I.

FUNDAMENTAL GEOMETRICAL PROPERTIES OF A LINEAR QUATERNION FUNCTION.

1. The quaternion equation

$$f(p+q)=fp+fq \quad \text{. . . . .} \quad (1),$$

may be regarded as a definition of the nature of a linear quaternion function f, the quaternions p and q being perfectly arbitrary. As a corollary, if x is any scalar,

$$f(xp)=xfp \quad \text{. . . . .} \quad (2),$$

and on resolving fq in terms of any four arbitrary quaternions a_1, a_2, a_3, a_4, we must have an expression of the form

$$fq=a_1Sb_1q+a_2Sb_2q+a_3Sb_3q+a_4Sb_4q \quad \text{. . . . . .} \quad (3),$$

because the coefficients of the four quaternions a must be scalar and distributive functions of q. Sixteen constants enter into the composition of the function f, being four for each of the quaternions b.

2. When a quaternion is regarded as the symbol of a point, the operation of the function f produces a linear transformation of the most general kind.

The equations

$$f(xa+yb)=xfa+yfb\,;\quad f(xa+yb+zc)=xfa+yfb+zfc \quad \text{. .} \quad (4),$$

show that the right line a, b is converted into the right line fa, fb, and the plane containing three points a, b, c into the plane containing their correspondents, fa, fb and fc.

The homographic character of the transformation is also clearly exhibited.

3. In order to specify a function of this kind it is necessary to know the quaternions a', b', c', d' into which any set of four unconnected quaternions, a, b, c, d, are converted. Thus, from the identical relation

$$q(abcd) + a(bcdq) + b(cdqa) + c(dqab) + d(qabc) = 0 \quad . \quad . \quad . \quad (5),$$

connecting one arbitrary quaternion with the four given quaternions, is deduced the equation

$$fq(abcd) + a'(bcdq) + b'(cdqa) + c'(dqab) + d'(qabc) = 0 \quad . \quad . \quad . \quad (6),$$

which determines the result of operating by f on q.

When we are merely concerned with the geometrical transformation of points, the absolute magnitudes* of the representative quaternions cease to be of importance, and the function

$$fq = x\mathrm{A}'(\mathrm{BCD}q) + y\mathrm{B}'(\mathrm{CD}q\mathrm{A}) + z\,\mathrm{C}'(\mathrm{D}q\mathrm{AB}) + w\mathrm{D}'(q\mathrm{ABC}) \,. \quad . \quad . \quad . \quad (7),$$

which involves four arbitrary scalars, converts the four points A, B, C, D into four others, A′, B′, C′, ′D. Given a fifth point E and its correspondent E′, the four scalars are determinate to a common factor, and subject to a scalar multiplier, the function which produces the transformation is

$$fq = \mathrm{A}'(\mathrm{BCD}q)\,.\,\frac{(\mathrm{B'C'D'E'})}{(\mathrm{BCDE})} + \mathrm{B}'(\mathrm{CD}q\mathrm{A})\,.\,\frac{(\mathrm{C'D'E'A'})}{(\mathrm{CDEA})} + \mathrm{C}'(\mathrm{D}q\mathrm{AB})\,.\,\frac{(\mathrm{D'E'A'B'})}{(\mathrm{DEAB})}$$
$$+ \mathrm{D}'(q\mathrm{ABC})\,.\,\frac{(\mathrm{E'A'B'C'})}{(\mathrm{EABC})} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (8).$$

It is only necessary to replace q by E in order to verify this result.

4. *A linear quaternion function, f, being regarded as effecting a transformation of points, the inverse of its conjugate f'^{-1} produces the corresponding tangential transformation.*

For any two quaternions, p and q,

$$\mathrm{S}pq = \mathrm{S}pf^{-1}q' = \mathrm{S}f'^{-1}pq' = \mathrm{S}p_{\prime}q' \text{ if } q' = fq,\ p_{\prime} = f'^{-1}p \quad . \quad . \quad . \quad (9).$$

Hence any plane $\mathrm{S}pq = 0$, in which the quaternion q represents the current point, transforms into the plane $\mathrm{S}p_{\prime}q' = 0$, and the proposition is proved.

Thus, when symbols of points (q) are transformed by the operation of f, symbols of planes (p), or of points reciprocal to the planes, are transformed by the operation of f'^{-1}.

5. HAMILTON's beautiful method of inversion of a linear quaternion function receives a geometrical interpretation from the results of the last article.

* In accordance with the notation proposed ('Trans. Roy. Irish Acad.,' vol. 32, p. 2), capital letters are used in this article concurrently with small letters to denote the same points, but the weights for the capital symbols are unity; thus $q = \mathrm{Q}\mathrm{S}q = (1 + o\mathrm{Q})\,\mathrm{S}q$.

The symbol of the plane containing three points a, b, c, may be written in the form

$$p = [a, b, c] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10);$$

and on transformation this becomes

$$np_{,} = [fa, fb, fc] = F'[abc] = F'p = F'f'p_{,} \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

where n is a certain scalar and where F' is an auxiliary function.

In fact, the first equation sums up the last article; in the second a new function F' is introduced, and in the fourth equation (9) is utilized.

Since $p_{,}$ is quite arbitrary (11) may be replaced by the symbolical equations

$$n = F'f'\,;\ F' = nf'^{-1}\,;\ f' = n^{-1}F'^{-1}\,;\ n = f'F' \quad . \quad . \quad . \quad . \quad (12),$$

an arbitrary quaternion being understood as the subject of the operations.

Moreover, because

$$n\mathrm{S}pq = \mathrm{S}pF'f'q = \mathrm{S}Fpf'q = \mathrm{S}qfFp \quad . \quad . \quad . \quad . \quad . \quad . \quad (13),$$

where p and q are arbitrary quaternions and where F is the conjugate of F', it appears that

$$n = fF\,;\ F = nf^{-1}\,;\ f = n^{-1}F^{-1}\,;\ n = Ff \quad . \quad . \quad . \quad . \quad . \quad . \quad (14).$$

And for any three arbitrary quaternions

$$F[abc] = [f'af'bf'c] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

as appears from symmetry, or, anew geometrically, by considering a point as the intersection of three planes.

Operating on the last equation by $\mathrm{S}f'd$ we find, since $n = fF = f'F'$,

$$n\,(abcd) = (f'af'bf'cf'd) = (fafbfcfd) \quad . \quad . \quad . \quad . \quad . \quad . \quad (16).$$

The fact that $(abcd)$ is a combinatorial function of a, b, c and d proves that n is an invariant, or that it is quite independent of any particular set of quaternions, a, b, c, d. This invariance is, however, established by the form of the equations (12) and (14).

6. Replacing f by $f_t = f - t$, where t is an arbitrary scalar, HAMILTON denotes by F_t and n_t the auxiliary function and the invariant which bear the same relations to f_t that F and n bear to f.

By (15) and (16), F_t and n_t are of the forms

$$F_t = F - tG + t^2H - t^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (17),$$

where G and H are new auxiliary functions; and

$$n_t = n - tn' + t^2n'' - t^3n''' + t^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18),$$

where n', n'' and n''' are new invariants.

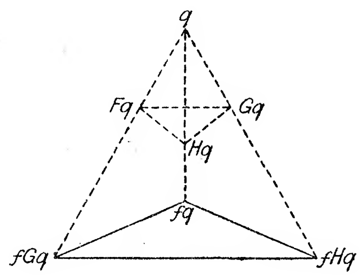
He then equates the coefficients of the arbitrary scalar t in the symbolical equation

$$n_t = f_t F_t = F_t f_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (19),$$

and obtains the symbolical equations

$$n = Ff,\ n' = F + Gf,\ n'' = G + Hf,\ n''' = H + f \quad . \quad . \quad . \qquad (20),$$

which will be found to be of great importance in the geometrical theory.



In virtue of (19), all these functions are commutative, in order of operation.

These equations establish certain collineations which are illustrated in the annexed figure.

From the relations (20) HAMILTON deduces

$$H = n''' - f;\ G = n'' - n'''f + f^2;\ F = n' - n''f + n'''f^2 - f^3 \quad . \quad . \qquad (21),$$

and the symbolic quartic satisfied by f

$$f^4 - n'''f^3 + n''f^2 - n'f + n = 0 \text{ or } (f - t_1)(f - t_2)(f - t_3)(f - t_4) = 0 \quad . \qquad (22),$$

if t_1, t_2, t_3, t_4 are the roots of the quartic

$$t^4 - n'''t^3 + n''t^2 - n't + n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (23),$$

or the *latent* roots of the function f.

It appears by (12) and (14) that exactly similar equations are valid for the conjugate function f', it being only necessary to replace F, G and H by their conjugates F', G' and H', as the invariants n, n', n'' and n''' are the same in both cases.

7. The united points of the transformation are represented by the quaternions q_1, q_2, q_3 and q_4, which satisfy the equations

$$fq_1 = t_1q_1;\ fq_2 = t_2q_2;\ fq_3 = t_3q_3;\ fq_4 = t_4q_4 \quad . \quad . \quad . \quad . \quad . \quad . \qquad (24);$$

and they are determined by operating on an arbitrary quaternion by the function obtained by omitting one factor of the second form of (22). In like manner by omitting two or three factors of the same quartic, the equations of the lines joining two, and of the planes through three, of the united points are obtained by operating on a variable quaternion. Thus

$$q = (f - t_3)(f - t_4)r \quad \text{and} \quad q = (f - t_4)r \quad . \quad . \quad . \quad . \quad . \quad . \qquad (25)$$

are respectively the equation of the line through the points q_1 q_2 and of the plane through the points q_1, q_2, q_3. These results are obvious when the arbitrarily variable point is referred to the united points as points of reference, or when we write

$$r = x_1q_1 + x_2q_2 + x_3q_3 + x_4q_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (26).$$

8. *The united points of a function and of its conjugate form reciprocal tetrahedra with respect to the unit sphere* $\mathrm{S}q^2 = 0$.

For when the roots are all unequal

$$t_1\mathrm{S}q'_2q_1 = \mathrm{S}q'_2fq_1 = \mathrm{S}q_1f'q'_2 = t_2\mathrm{S}q_1q'_2 = 0 \quad . \quad . \quad . \quad . \quad . \qquad (27),$$

if q'_1 q'_2, q'_3 and q'_4 are the united points of the conjugate. Thus the points q_1 and q'_2 are conjugate with respect to the sphere.

Since the plane $\mathrm{S}q'_1q = 0$ contains the points q_2, q_3, q_4, the weights may be chosen so that

$$q'_1 = \frac{[q_2q_3q_4]}{(q_1q_2q_3q_4)},\ q'_2 = \frac{[q_3q_4q_1]}{(q_2q_3q_4q_1)},\ q'_3 = \frac{[q_4q_1q_2]}{(q_3q_4q_1q_2)},\ q'_4 = \frac{[q_1q_2a_3]}{(q_4q_1q_2q_3)} \quad . \quad . \qquad (28);$$

and these relations imply*

$$\mathrm{S}q_1q'_1 = \mathrm{S}q_2q'_2 = \mathrm{S}q_3q'_3 = \mathrm{S}q_4q'_4 = 1 \qquad . \quad . \quad . \quad . \quad . \qquad (29);$$

and from symmetry

$$q_1 = \frac{[q'_2q'_3q'_4]}{(q'_1q'_2q'_3q'_4)},\ q_2 = \frac{[q'_3q'_4q'_1]}{(q'_2q'_3q'_4q'_1)},\ q_3 = \frac{[q'_4q'_1q'_2]}{(q'_3q'_4q'_1q'_2)},\ q_4 = \frac{[q'_1q'_2q'_3]}{(q'_4q'_1q'_2q'_3)}. \qquad (30).$$

To these relations may be added the quaternion identities

$$q_1q'_1 + q_2q'_2 + q_3q'_3 + q_4q'_4 = 4 = q'_1q_1 + q'_2q_2 + q'_3q_3 + q'_4q_4 \quad . \quad . \quad . \quad . \qquad (31),$$

$$q_1\mathrm{S}q'_1 + q_2\mathrm{S}q'_2 + q_3\mathrm{S}q'_3 + q_4\mathrm{S}q'_4 = 1 = q'_1\mathrm{S}q_1 + q'_2\mathrm{S}q_2 + q'_3\mathrm{S}q_3 + q'_4\mathrm{S}q_4 \ . \qquad (32),$$

which are probably more elegant than important. The second shows that the centre of the sphere is the centre of mass of the weights $\mathrm{S}q_1\mathrm{S}q'_1$, $\mathrm{S}q_2\mathrm{S}q'_2$, $\mathrm{S}q_3\mathrm{S}q'_3$, $\mathrm{S}q_4\mathrm{S}q'_4$ placed at the vertices of either of the tetrahedra, and that the sum of their weights is unity.

From these identities we deduce the vector equations

$$(q_1q'_1) + (q_2q'_2) + (q_3q'_3) + (q_4q'_4) = 0 = [q_1q'_1] + [q_2q'_2] + [q_3q'_3] + [q_4q'_4] \qquad (33),$$

which express that equilibrating forces can be placed along the lines joining corresponding vertices, or that any line which meets three of these lines meets the fourth, or that the lines are generators of a quadric.†

* Writing $q_1 = w_1(1 + \alpha_1)$, $q'_1 = w'_1(1 + \alpha'_1)$, equations (29) give $w_1w'_1(1 + \mathrm{S}\alpha_1\alpha'_1) = 1$. Hence the product of the weights $w_1w'_1$ is the reciprocal of the product of the perpendiculars from the centre of the sphere and from the point q_1 (or q'_1) on the opposite face of the tetrahedron $q_1q_2q_3q_4$ (or $q'_1q'_2q'_3q'_4$). Observe that only the products $w_1w'_1$ have been assigned, not w_1 and w'_1 separately.

† In the notation of the last note (33) becomes $\Sigma w_1w'_1(\alpha'_1 - \alpha_1) = \Sigma w_1w'_1\mathrm{V}\alpha_1\alpha'_1 = 0$. The equilibrating forces are proportional to the distances between the vertices divided by the products of perpendiculars mentioned in the note cited.

It is also possible to obtain relations connecting pairs of the points ((N), p. 224),

$$[q_1q_2] = +\frac{(q'_3q'_4)}{(q'_1q'_2q'_3q'_4)};\ (q_1q_2) = -\frac{[q'_3q'_4]}{(q'_1q'_2q'_3q'_4)};\ [q'_1q'_2] = +\frac{(q_3q_4)}{(q_1q_2q_3q_4)};$$
$$(q'_1q'_2) = -\frac{[q_3q_4]}{(q_1q_2q_3q_4)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

from which we learn that

$$-1 = (q_1q_2q_3q_4)(q'_1q'_2q'_3q'_4) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (35);$$

and we are at liberty to write separately on further selection of the weights (for the products of the weights $Sq_1Sq'_1$ alone have been assigned),

$$(q_1q_2q_3q_4) = (q'_1q'_2q'_3q'_4) = \sqrt{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (36),$$

with corresponding simplifications in the formulæ.

When the function is self-conjugate, the tetrahedron of united points is self-reciprocal to the unit sphere.

9. Introducing two new linear functions defined by the equations

$$f = f_0 + f_{\prime},\ f' = f_0 - f_{\prime},\ \text{or } 2f_0 = f + f',\ 2f_{\prime} = f - f' \quad . \quad . \quad . \quad . \quad (37),$$

it is obvious that for any two quaternions, p and q,

$$Spf_0q = Sqf_0p;\ Spf_{\prime}q = -Sqf_{\prime}p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38),$$

or symbolically

$$f_0 = f_0',\quad f_{\prime} = -f_{\prime}' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (39),$$

and f_0 is self-conjugate, and $f_{\prime}$ is the negative of its conjugate.

10. The equation

$$Sqf_0q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40)$$

is the general equation of a quadric surface, and

$$Sqf_{\prime}p = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (41)$$

is that of a linear complex, p and q being both variable points.

In fact (40) is the most general scalar quadratic function homogeneous in q, and the surface represented meets the arbitrary line $q = a + tb$ in the points determined by the roots of the quadratic

$$Saf_0a + 2tSaf_0b + t^2Sbf_0b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (42).$$

In like manner (41) is the most general scalar function linear in two quaternions and combinatorial with respect to both, for by (38)

$$Sqf_{\prime}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (43)$$

whatever quaternion q may be. It is therefore immaterial if we replace q and p in (41) by any other points on their line, provided the two points are not coincident, and

the equation therefore imposes a single linear restriction on the line pq, and represents a linear complex.

In terms of vectors, putting $q = 1 + \rho$, $p = 1 + \varpi$, and using the expression given in the 'Elements' (Art. 364, XII.) for a linear quaternion function, we have

$$fq = e + \epsilon + S\epsilon'\rho + \phi\rho,\ f'q = e + \epsilon' + S\epsilon\rho + \phi'\rho;$$

$$f_0 q = e_0 + \epsilon_0 + S\epsilon_0\rho + \phi_0\rho,\ f_{,}q = \epsilon_{,} - S\epsilon_{,}\rho + V\eta\rho \quad . \quad . \quad . \quad (44);$$

where

$$e_0 = e,\ 2\epsilon_0 = \epsilon + \epsilon',\ 2\epsilon_{,} = \epsilon - \epsilon';\ \phi = \phi_0 + V\eta,\ \phi' = \phi_0 - V\eta,$$

and the equations of the quadric and linear complex assume well-known forms

$$e_0 + 2S\epsilon_0\rho + S\rho\phi_0\rho = 0,\ S\epsilon_{,}(\varpi - \rho) + S\eta V\rho\varpi = 0 \quad . \quad . \quad . \quad (45).$$

11. The equation of the polar plane of a point a with respect to the quadric (compare (42)) is

$$Sqf_0a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46),$$

and f_0a is the pole of this plane with respect to the unit sphere.

Thus f_0a is the symbol of the polar plane of the point a.

With respect to the quadric the pole of the plane

$$Sqb = 0 \text{ is } p = f_0^{-1}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (47),$$

and the reciprocal of the quadric has for its equation

$$Sqf_0^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (48).$$

The lines of the complex through a given point a lie in the plane

$$Sqf_{,}a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49),$$

while the point of concourse of the lines in the plane

$$Sqb = 0 \text{ is } p = f_{,}^{-1}b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (50),$$

and

$$Spf_{,}^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (51)$$

is the equation of the reciprocal of the complex.

12. The nature of the united points of the function $f_{,}$ is easily ascertained.

Since the function is the negative of its conjugate, its symbolic quartic (22) must be of the form

$$f_{,}^4 + n_{,}''f_{,}^2 + n_{,} = 0, \text{ or } (f_{,}^2 - s_1^2)(f_{,}^2 - s_2^2) = 0 \quad . \quad . \quad . \quad . \quad (52).$$

And if

$$f_{,}p_1 = s_1p_1,\ f_{,}p'_1 = -s_1p'_1,\ f_{,}p_2 = s_2p_2,\ f_{,}p'_2 = -s_2p'_2 \quad . \quad . \quad . \quad (53),$$

it follows in the first place (43) that the united points all lie on the unit sphere, and in the second by (27) that

$$Sp_1p_2 = Sp_1p'_2 = Sp'_1p_2 = Sp'_1p'_2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (54).$$

Hence in this order $p_1p_2p'_1p'_2$ is a quadrilateral situated on the unit sphere.

These results may be verified for the vector form (44). Actually solving

$$f_{,}(1 + \varpi) = s(1 + \varpi) = \epsilon_{,} - S\epsilon_{,}\varpi + V\eta\varpi,$$

we see that $s = -S\epsilon_{,}\varpi$, $sS\eta\varpi = S\eta\epsilon_{,}$, and therefore

$$(s - \eta)\varpi = \epsilon_{,} - s^{-1}S\eta\epsilon_{,}, \text{ or } (s^2 - \eta^2)\varpi = (s + \eta)(\epsilon_{,} - s^{-1}S\eta\epsilon_{,}),$$

so that operating by $S\epsilon_{,}$ the result is the quartic in s

$$s^4 + s^2(\epsilon_{,}^2 - \eta^2) - (S\eta\epsilon_{,})^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (55);$$

and for a real function two roots of this quartic are always real and two are imaginary. Two of the united points are consequently real (Art. 7) and two are imaginary.

SECTION II.

The Classification of Linear Quaternion Functions.

13. Linear quaternion functions may be classified according to the nature of the united points :—

I. The first class consists of those functions which have no line or plane locus of united points, and it is divisible into sub-class :—

I_1, the four united points distinct.
I_2, two united points coincident.
I_3, three united points coincident.
I_4, all four coincident.
I_5, two distinct pairs of coincident united points.

II. The second class consists of functions having a line locus of united points, with the following sub-classes :—

II_1, the two remaining united points distinct.

II_2, the two remaining united points coincident.

II_3, one of the remaining united points on the line locus.

II_4, the two remaining points coincident and on the line locus.

III. The third class consists of functions having a plane locus of united points, and there are two sub-classes :—

III_1, the remaining united point is not in the plane.

III_2, the remaining united point is in the plane.

IV. The functions of the fourth class have two line loci of united points.

It is to be noticed that any peculiarity in a function is exactly reproduced in its conjugate. This will appear clearly from the following discussion, but the proposition is virtually proved in the concluding remarks of Art. 6.

To assist in the examination of the different cases, it is convenient to repeat HAMILTON's relations (20) and (21), and in addition to obtain the symbolic quartics for the function H, G, and F. These quartics are deducible from the relations (20) or (21) without much trouble. The group of formulæ is thus :—

$$Ff = n, \quad F + Gf = n', \quad G + Hf = n'', \quad H + f = n''' ;$$

$$H = n''' - f, \quad G = n'' - n'''f + f^2, \quad F = n' - n''f + n'''f^2 - f^3 ;$$

$$f^4 - n'''f^3 + n''f^2 - n'f + n = 0 \quad \ldots \ldots \quad (56).$$

$$H^4 - 3n'''H^3 + (n'' + 3n'''^2) H^2 + (n' - 2n''n''' - n'''^3) H + n - n'n''' + n''n'''^2 = 0.$$

$$G^4 - 2n''G^3 + (2n + n'n''' + n''^2) G^2 - (2nn'' - nn'''^2 + n'^2 + n'n''n''') G + n^2 - nn'n''' + n'^2n'' = 0.$$

$$F^4 - n'F^3 + nn''F^2 - n^2n'''F + n^3 = 0.$$

14. For the sake of brevity in discussing the various classes, one root of the scalar quartic is supposed to be reduced to zero by replacing the function by one of the four functions $f - t_1, f - t_2, f - t_3, f - t_4$ of Art. 6 ; and whenever there is a multiple root, it is the multiple root which is reduced to zero.

I. *One quaternion, "a," is reduced to zero by the operation of the function.*

Remembering that the conjugate also reduces a quaternion a' to zero, it follows if

$$fa = 0, f'a' = 0 \quad \ldots \ldots \quad (57)$$

that the locus of the transformed points, $p = fq$, is a fixed plane,

$$Spa' = 0 \quad \ldots \ldots \quad (58),$$

because $Sqf'a' = 0$. Every plane through the point a is reduced to a line; every line through the point becomes a point; the scalar n is zero; the function F reduces every point to a and destroys every point in the fixed plane (58).

The quadrinomial (3) must reduce to a trinomial, for f cannot destroy a quaternion unless there is a relation between a_1, a_2, a_3, a_4, or else between b_1, b_2, b_3, b_4. The type of functions of this kind is

$$fq = a_1 Sa'_1 q + a_2 Sa'_2 q + a_3 Sa'_3 q\,;\ a = [a'_1 a'_2 a'_3],\ a' = [a_1 a_2 a_3] \quad . \quad . \quad (59).$$

II. *The function destroys two distinct points.*

If

$$fa = 0, fb = 0\,;\ f'a' = 0, f'b' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (60)$$

the line a, b is destroyed. The locus of the transformed points is the line of intersection of the planes

$$Spa' = 0,\ Spb' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61).$$

Every plane and every line through the line a, b is reduced to a point. The function is reducible to the binomial type

$$fq = a_1 Sa'_1 q + a_2 Sa'_2 q\,;\ a + tb = [a'_1 a'_2 r'],\ a' + t'b' = [a_1 a_2 r] \quad . \quad . \quad . \quad . \quad (62),$$

when r and r' are quite arbitrary, and it is evident (15) that the function F vanishes identically.

III. *The function destroys three non-collinear points.*

$$fa = 0, fb = 0, fc = 0\,;\ f'a' = 0, f'b' = 0, f'c' = 0 \quad . \quad . \quad . \quad . \quad (63)\,;$$

and every point is reduced to a fixed point, the intersection of the planes

$$Spa' = 0,\ Spb' = 0,\ Spc' = 0, \text{ or } p = [a'b'c'] \quad . \quad . \quad . \quad . \quad (64).$$

Hence the function is a monomial,

$$fq = [a'b'c']\, S\, [abc] q = a_1 Sa'_1 q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (65),$$

and the function G vanishes identically.

IV. *The function destroys two distinct points, a and b, and alters the weights of two others, c and d, in the same ratio, but otherwise leaves these points unchanged.*

The type is

$$fq\,.\,(abcd) = t_0 c(abqd) + t_0 d(abcq) \quad . \quad . \quad . \quad . \quad . \quad . \quad (66).$$

15. In order to illustrate the nature of the solution of the equation

$$fq = p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (67)$$

in the different cases, we employ HAMILTON's relations (56), which give the solution on substitution.

I_1. *One latent root is zero.* In this case

$$n = 0, Fp = 0\,; \quad n'q = Gp + Fq = Gp + xa \quad . \quad . \quad . \quad . \quad . \quad (68),$$

because F reduces every quaternion to the fixed quaternion a multiplied by a scalar x. Here x is arbitrary, provided the condition $Fp = 0$ is satisfied; the point Gp lies in the fixed plane (58); and q may be any point on the line joining this point to a, or in other words, this line is the solution of the equation (67).

If the condition $Fp = 0$ is not satisfied, the scalar x must be infinite, so that in the limit $f(Gp + xa)$ may have a component at the point a, which escapes destruction by F. The solution is simply the point a affected with an infinite weight.

When $n = 0$, it appears from HAMILTON's relations that F satisfies the depressed equation

$$F(F - n') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69),$$

and the interpretation is, F reduces an arbitrary quaternion to a; $F - n'$ destroys a.

I_2. *Two latent roots are zero.* Here

$$n = n' = 0, \quad Fp = 0\,; \quad Gp + Fq = 0\,; \quad n''q = Hp + Gq \quad . \quad . \quad (70),$$

and q must be allowed the full extent of arbitrariness consistent with the conditions.

Observing that the relations (56) now give

$$f^2G = 0, \quad Gf^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (71)$$

it appears that the double operation of f destroys the result of operating on any quaternion by G, and that G destroys f^2q. Hence,

$$Gq = xa' + ya, \quad \text{where} \quad fa' = a, f^2a' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (72).$$

The scalar x is determinate for

$$fGq = Gp = xa \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (73),$$

but y is arbitrary, and the solution is any point on the line, y variable,

$$n''q = Hp + xa' + ya \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (74).$$

As before, if Fp is not zero, the solution is a multiplied by an infinite scalar.

The character of the function G has now completely changed. It now destroys a line (f^2q), and because $Gf^2 = 0$, or $Gf^2H^2 = 0$, and also $n' = 0$, the symbolic equations of G and F are both degraded, and are

$$G(G - n'')^2 = 0, \quad F^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (75).$$

I_3. The solution in this case is

$$n = n' = n'' = 0\,; \quad Fp = 0, \quad Gp + Fq = 0, \quad Hp + Gq = 0\,; \quad n'''q = p + Hq \quad . \quad (76).$$

The symbolic equations now give

$$F^2 = 0, \quad G^3 = 0, \quad Hf^3 = 0, \quad G = -Hf, \quad F = Hf^2 \quad . \quad . \quad . \quad (77).$$

and

$$Hq = xa'' + ya' + za \quad \text{where} \quad fa'' = a', \quad fa' = a, \quad fa = 0 \quad . \quad . \quad (78).$$

The solution thus takes the more explicit form,

$$n'''q = p + xa'' + ya' + za\,; \quad Hp = xa' + ya\,; \quad Gp = -xa, \quad Fp = 0\,. \quad (79),$$

and z alone is arbitrary.

If the last condition is not fulfilled, z is infinite.

I_4. Again, where $n''' = 0$, the solution is any point on the line, w variable,

$$q = xa''' + ya'' + za' + wa\,;\; p = xa'' + ya' + za\,;\; fp = ya' + ya\,;\; f^2p = xa\,;\; f^3p = 0\,. \quad (80).$$

The symbolical equations satisfied by F, G, H and f are now

$$F^2 = 0, \quad G^2 = 0, \quad H^4 = 0, \quad f^4 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (81).$$

Although the forms of the equations for F and G are identical, the nature of these functions are widely different; G reduces an arbitrary point to the line $xa' + ya$, which is destroyed by a further application of the same function; F reduces an arbitrary point at once to the point wa, which is destroyed by a successive operation.

The type of a function of this class I_4 is

$$fq\,(aa'a''a''') = a\,(aqa''a''') + a'\,(aa'qa''') + a''\,(aa'a''q) \quad . \quad . \quad . \quad (82),$$

in which a, a', a'' and a''' are arbitrary quaternions.

The function,

$$f(q)\,.\,(aa'bb') = a\,(aqbb') + t_0 b\,(aa'qb') + (b + t_0 b')\,(aa'bq)\,. \quad . \quad . \quad (83)$$

belongs to the sub-class I_5.

16. II_1. A function of the second class destroys two points, a and b, and in virtue of the distributive property it destroys the line a, b.

Since the locus of fq is a line (61), the function F vanishes identically (15), and likewise the invariant n' as well as n.

HAMILTON's relations become,

$$n = n' = 0\,; \quad F = 0, \quad Gf = 0\,; \quad Hf + G = n'', \quad H + f = n''' \quad . \quad . \quad (84)$$

and the symbolic equations for f and G degrade into

$$F = f^3 - n'''f^2 + n''f = 0\,; \quad G\,(G - n'') = 0 \quad . \quad . \quad . \quad . \quad (85).$$

The function $G - n''$ destroys the line a, b, which is consequently the locus of Gq.

For the solution of the equation $fq = p$, the relations (84) give

$$n''q = Hp + Gq\,; \quad Gp = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (86);$$

and since Gq may be any point on the line a, b, the locus of q is the plane $[Hp, a, b]$.

If $Gp = 0$ is not satisfied, the solution is an arbitrary point on the line a, b affected with an infinite weight.

II$_3$. If $n'' = 0$, the solution is

$$n'''q = p + Hq\,; \quad Hp + Gq = 0, \quad Gp = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (87),$$

and

$$G^2 = 0, \quad Hf^2 = 0, \quad Hf = -G \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (88),$$

whence

$$Hq = xa' + ya + zb, \quad Hp = xa \quad \text{if} \quad a' = fa \quad . \quad . \quad . \quad . \quad . \quad (89).$$

II$_4$. If further, $n''' = 0$, the solution is

$$q = xa'' + ya' + za + wb, \quad p = xa' + ya, \quad fp = xa, \quad f^2p = 0 \quad . \quad . \quad (90);$$

and the general function of this type is

$$fq\,(aba'a'') = a\,(abqa'') + a'\,(aba'q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91),$$

and the function G of I$_4$ is of this sub-class.

17. III$_1$. The third class is that in which f destroys three points a, b, c, which are not situated on a common line

Here

$$n = n' = n'' = 0\,; \quad F = G = Hf = 0\,; \quad n''' = f + H, \quad f^2 - n'''f = 0 \quad . \quad (92),$$

and the solution is

$$n'''q = p + xa + yb + zc \quad \text{where} \quad Hp = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (93).$$

III$_2$. If $n''' = 0$,

$$q = xa' + ya + zb + wc \quad \text{where} \quad p = xa, \quad fa' = a \quad . \quad . \quad . \quad . \quad (94).$$

The type of the function is

$$f(q)\,.\,(abca') = a\,(abcq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (95),$$

to which the function F of I$_4$ belongs.

18. IV. The fourth class is that in which two lines ab and cd are destroyed.

$$n = n' = 0, \quad F = 0, \quad Gf = 0, \quad Hf + G = n'' = \tfrac{1}{4}n'''^2, \quad H + f = n''' \quad . \quad (96)$$

and the symbolic equations are

$$f\,(f - \tfrac{1}{2}n''') = 0\,; \quad G\,(G - n'') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (97).$$

The function

$$f(q)\,.\,(abcd) = t_0c(abqd) + t_0d\,(abcq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (98)$$

is of this type. f destroys the line a, b and reduces an arbitrary point to c, d; $f - t_0$ destroys c, d and reduces an arbitrary point to ab.

19. As the theory of the self-conjugate linear vector function differs in various details from that of the self-conjugate quaternion function, it is necessary to devote a few remarks to the latter.

The four united points of a self-conjugate function form a tetrahedron self-conjugate to the unit sphere, for in this case the two tetrahedra of Art. 8 coincide. If two united points coincide, they must coincide with a point on the sphere, and the scalar quartic has a pair of equal roots. But in the case of a *real* self-conjugate vector function when two latent roots are equal, the function has an infinite number of axes in a certain plane, and not a single axis resulting from the coalescence of a pair; and the reason is simply that a real vector cannot be perpendicular to itself, while each axis of a self-conjugate vector function must be perpendicular to two others. For a quaternion function, on the other hand, a real point may be its own conjugate with respect to the unit sphere, and there may be in this case coincidence of united points without a locus of united points and consequent degradation of the symbolic quartic.

Again, the roots and axes of a self-conjugate vector function must be real, because two conjugate imaginary vectors, $\alpha + \sqrt{-1}\,\beta$, $\alpha - \sqrt{-1}\,\beta$, cannot be at right angles to one another, since the condition is $\alpha^2 + \beta^2 = 0$, while $\alpha^2 + \beta^2$ is essentially negative. But two united points of a real self-conjugate quaternion function may be conjugate imaginaries, the condition

$$S\,(a + \sqrt{-1}b)\,(a - \sqrt{-1}b) = Sa^2 + Sb^2 = 0 \quad . \quad . \quad . \quad (99),$$

merely showing that the real points a and b are situated one inside and one outside the unit sphere.

20. On account of the importance of the self-conjugate function, it may not be superfluous to illustrate cases of coalesced united points.

Writing for the general self-conjugate function,

$$f(1+\rho) = e + \epsilon + S\epsilon\rho + \Phi\rho\,;\ \ S\,(1+\rho)\,f(1+\rho) = e + 2S\epsilon\rho + S\rho\Phi\rho \quad . \,(100),$$

the latent quartic is

$$t^4 - t^3\,(e + m'') + t^2\,(em'' + m' - \epsilon^2) - t\,(em' + m + S\epsilon\,(\Phi - m'')\,\epsilon)$$
$$+\, m\,(e - S\epsilon\Phi^{-1}\epsilon) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (101).$$

The quadric surface $Sqfq = 0$ has its centre at the extremity of the vector $-\Phi^{-1}\epsilon$, or say at the point c.

One root is zero if

$$e - S\epsilon\Phi^{-1}\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (102),$$

and the quadric is a cone with its vertex at the point c. A second root is zero if

$$m = -S\epsilon\,(\Phi - m'' + m'\Phi^{-1})\,\epsilon = -mS\epsilon\Phi^{-2}\epsilon, \text{ or if } T\Phi^{-1}\epsilon = 1 \quad . \quad . \quad (103)\,;$$

that is, if the vertex is on the unit sphere.

A third root is zero if

$$m' = S\epsilon\,(1 - m''\Phi^{-1})\,\epsilon, \text{ or if } S\epsilon\,(1 - m''\Phi^{-1} + m'\Phi^{-2})\,\epsilon = mS\epsilon\Phi^{-3}\epsilon = 0 \quad . \quad (104),$$

and this simply requires $\Phi^{-2}\epsilon$ to be parallel to a generator of the cone, and perpendicular to the vector to its vertex. This generator touches the sphere.

The condition that the fourth root may vanish reduces to

$$mT\Phi^{-2}\epsilon = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (105),$$

and requires $m = 0$ for a real function, and in this case the cone breaks into a pair of planes, and the symbolic quartic degrades.

Admitting that $T\Phi^{-2}\epsilon = 0$ (for an imaginary function), it appears that the generator $-\Phi^{-1}\epsilon + x\Phi^{-2}\epsilon$ is common to the quadric and the sphere when four roots are zero.

The preceding analysis establishes the fact that a real self-conjugate function may belong to the classes, I_1, I_2, I_3, II_4 but not to I_4.

A real self-conjugate function cannot belong to I_5 if its two united points are real, for certain of the conditions of self-conjugation of the tetrahedron in the limit require $Sa^2 = Sab = Sb^2 = 0$, or the line a, b must be a generator of the sphere; and matters are not changed when we assume a and b to be conjugate imaginaries. We conclude therefore that no self-conjugate function belongs to I_5.

Since self-conjugate functions of the type II_4 exist, *à fortiori* they will exist for the less restricted types II_1, II_2, II_3.

Self-conjugate functions may belong to the types III_1, III_2, and to type IV, the lines being now conjugate with respect to the sphere (compare the following Article).

21. *If a function converts any tetrahedron into its reciprocal, it is self-conjugate.*

Here if

$$fa = x\,[bcd], \quad fb = y\,[acd], \quad fc = z\,[abd], \quad fd = w\,[abc] \quad . \quad . \quad (106),$$

the function producing the transformation is

$$fq\,(abcd) = x\,[bcd]\,(qbcd) - y\,[acd]\,(qacd) + z\,[abd]\,(qabd) - w\,[abc]\,(qabc) \quad . \quad . \quad . \quad (107),$$

which is manifestly self-conjugate.

This includes as a particular case the deduction from Art. 8.

The following theorems may be stated here :—

If a function has a scalar for a principal solution, its conjugate has three vector principal solutions.

If a function has a line or a plane locus of united points, it has a vector or a linear system of vector principal solutions.

The nature of the function $f_{\prime}$, which is the negative of its conjugate, has been sufficiently considered in Art. 12.

22. It may be as well to show the geometrical meaning of changing from a function f to another $f - t_0$, as in Art. 14.

Writing

$$p' = (f - t_0)\, q = p - t_0 q, \quad p = fq \quad . \quad . \quad . \quad . \quad . \quad . \quad (108);$$

it is obvious that p' is some point on the line pq. To determine the point, let P′, P and Q be the points p', p and q with unit weights, then

$$\mathrm{P}' = \frac{p - t_0 q}{Sp - t_0 Sq} = \frac{f\mathrm{Q} - t_0\mathrm{Q}}{Sf\mathrm{Q} - t_0} = \frac{\mathrm{P}Sf\mathrm{Q} - t_0\mathrm{Q}}{Sf\mathrm{Q} - t_0} \quad . \quad . \quad . \quad . \quad (109);$$

and we have the ratio of segments

$$\frac{\mathrm{P'Q}}{\mathrm{P'P}} = \frac{\mathrm{Q} - \mathrm{P}'}{\mathrm{P} - \mathrm{P}'} = \frac{Sf\mathrm{Q}}{t_0} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (110),$$

or its ratio is directly proportional to the perpendicular from the point Q on the plane $Sfq = 0$, which is projected to infinity by the transformation.*

Hence it is easy to form a geometrical conception of the nature of a transformation by reducing it to some simpler type, as in Art. 14; the point P for instance may always be supposed to lie in a fixed plane, while in the case of functions of the classes II and III it may be supposed to lie on a fixed line or to be a fixed point.

SECTION III.

Scalar Invariants.

23. From the results of Arts. 5 and 6, it appears that

$$((f - t)\, a, (f - t)\, b, (f - t)\, c, (f - t)\, d) = (abcd)\, (n - n't + n''t^2 - n'''t^3 + t^4) \quad (111)$$

is identically true, no matter what the value of t may be or what quaternions a, b, c, and d may represent. In this sense the four scalars n, n', n'', and n''' are invariants, and every relation connecting them implies some peculiarity in the geometrical transformation produced by f.

* In vectors, if $\mathrm{Q} = 1 + \rho$, the ratio is $t_0^{-1}(e + S\epsilon'\rho) = t_0^{-1} x T\epsilon'$ if x is the length of the perpendicular.

But there is a wider sense in which these four scalars are invariants. If n_1 and n_2 are the fourth invariants of any two functions f_1 and f_2, the relation

$$((f_1ff_2 - tf_1f_2)\,a, (f_1ff_2 - tf_1f_2)\,b, (f_1ff_2 - tf_1f_2)\,c, (f_1ff_2 - tf_1f_2)\,d)$$
$$= (abcd)\,n_1n_2\,(n - n't + n''t^2 - n'''t^3 + t^4) \quad (112)$$

is evidently true or may be verified at once by repeated application of (16). Thus any relation implying a peculiarity of the function f and depending on its four invariants, implies also a corresponding peculiarity in the mutual relations of the functions f_1ff_2 and f_1f_2, that is, of any two functions F_1 and F_2 decomposible in the manner indicated. In particular, if in (112) f_2 is replaced by f_1^{-1}, it is evident that the invariants of $f_1ff_1^{-1}$ are identical with those of f. And, moreover, the functions may be replaced by their conjugates without altering the invariants.

We now propose to examine the meaning of a few invariants, bearing in mind the remarks of this article, and remembering also that the invariants are more general than those of quadrics, for the function f is not supposed to be self-conjugate.

24. For brevity, replacing fa by a', we have

$$n'''\,(abcd) = (a'bcd) + (ab'cd) + (abc'd) + (abcd') \quad . \quad . \quad . \quad . \quad (113).$$

If n''' vanishes, it is possible to determine an infinite number of tetrahedra a, b, c, d, so that the corners of a derived tetrahedron shall lie on the faces of the original.

For taking any three points a, b, c, and their deriveds a', b', c', three planes are found

$$(a'bcd) = 0, \quad (ab'cd) = 0, \quad (abc'd) = 0 \quad . \quad . \quad . \quad . \quad . \quad (114),$$

whose common point d enjoys the property of having its derived in the plane of a, b, and c if, and only if, $n''' = 0$.

Conversely, if this is true for any tetrahedron and its derived, the invariant n''' vanishes, and the property is true for an infinite number of tetrahedra.

Interchanging the words *corner* and *face*, we have the corresponding interpretation of the vanishing of n'.

More generally, when n''' vanishes, an infinite number of tetrahedra exists, so that the pairs derived from them by the operations of the functions f_1ff_2 and f_1f_2 are related in the manner described.

Analogous extensions will be understood in the sequel.

25. Again, suppose that the sum of the squares of the roots of $n_t = 0$ is zero, or that

$$n'''^2 - 2n'' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

In this case, tetrahedra may be found related to their correspondents in such a manner that the deriveds of these correspondents have their corners on the faces of the originals.

Of greater interest, however, is the case in which the sum of the square roots of the roots of $n_t = 0$ is zero, or when

$$(n'''^2 - 4n'')^2 = 64n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

Here the n''' invariant of one of the square roots of the function (compare Art. 36) vanishes, so that by the operation of this square root $f^{\frac{1}{2}}$, it is possible, from a suitably selected tetrahedron (one of an infinite number), to derive a second, and from that again a third, so that the second has its corners on the faces of the first, while its faces contain the corners of the third. But directly by the operation of $f (= f^{\frac{1}{2}} . f^{\frac{1}{2}})$ the third tetrahedron is transformed from the first, and these are so related that it is possible to inscribe to the first a tetrahedron circumscribed to the third.

Similarly, we can interpret invariants arising from relations such as

$$t_1^m + t_2^m + t_3^m + t_4^m = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (117),$$

where m is the ratio of two integers, and where t_1, t_2, t_3, and t_4 are the latent roots of f.

26. Before passing on to invariants of a rather different type, we shall consider the relation connecting two quadric surfaces when an *infinite* number of tetrahedra can be inscribed to one and circumscribed to another.

Let the equations of the quadrics be

$$SqF_1q = 0, \quad SqF_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (118);$$

let the tetrahedron $(abcd)$ be inscribed to the first, and let its faces touch the second at the points a', b', c', d'; let the function f derive the tetrad of points of contact from the corresponding vertices. Then there are four equations of inscription to the first quadric

$$SaF_1a = 0, \quad SbF_1b = 0, \quad ScF_1c = 0, \quad SdF_1d = 0 \quad . \quad . \quad . \quad (119);$$

twelve equations of conjugation of the points a', b, &c., to the second quadric

$$Sa'F_2b = Sb'F_2a = 0 \quad \text{or} \quad Saf'F_2b = SaF_2fb = 0 \quad . \quad . \quad . \quad (120);$$

and four equations of contact such as

$$Sa'F_2a' = 0 \quad \text{or} \quad Saf'F_2fa = 0 .$$

The equations of conjugation require the function F_2f to be self-conjugate, so that

$$f'F_2 = F_2f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121),$$

and the conditions of contact may therefore be replaced by four equations such as

$$SaF_2f^2a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (122).$$

An infinite number of tetrahedra may consequently be respectively inscribed and circumscribed to the quadrics

$$SqF_2f^2q = 0, \quad SqF_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (123),$$

when the condition (121) is satisfied and when the n''' of f vanishes; and if this is likewise possible for the given quadrics, we must have

$$F_2f^2 = F_1, \quad \text{or} \quad f^2 = F_2^{-1}F_1, \quad \text{or} \quad f = (F_2^{-1}F_1)^{\frac{1}{2}} \quad . \quad . \quad . \quad (124).$$

The sum of the square roots of the latent roots of the function $F_2^{-1}F_1$ must consequently vanish, or the invariant* (116) of this function is zero.

It has been proved incidentally in this article if a tetrahedron circumscribed to $SqF_2q = 0$ is self-conjugate to $SqF_3q = 0$, that the invariant n''' of the function $F_2^{-1}F_3$ is zero; and if the tetrahedron is self-conjugate to $SqF_3q = 0$ and inscribed to $SqF_1q = 0$, that the same invariant of the function $F_3^{-1}F_1$ is zero. Here $F_3 = F_2f$.

It must be carefully observed that in dealing with quadrics the extent of the invariance (Art. 23) is limited. If F_1 and F_2 are self-conjugate, the functions $f_1F_1f_2$ and $f_1F_2f_2$ must be self-conjugate before theorems can be extended from the quadrics determined by the simpler to those determined by the more complex functions.

27. The invariant n'' vanishes if

$$(a'b'cd) + (a'bc'd) + (a'bcd') + (ab'c'd) + (ab'cd') + (abc'd') = 0 \quad . \quad (125).$$

To save verbiage in the interpretation, the edges ab and $c'd'$ may be called the *opposite* edges of a tetrahedron and its derived. If each edge of $(abcd)$ intersects the opposite edge of $(a'b'c'd')$, the invariant will manifestly vanish, for every term will be zero.

To display the nature of the conditions requisite for determining a tetrahedron possessing this property, when $n'' = 0$, let a and b be assumed fixed, and then five of the terms may be written in forms

$$Scf_nd = 0 \qquad (n = 1, 2, 3, 4 \text{ or } 5) \quad . \quad . \quad . \quad . \quad . \quad (126),$$

where f_n is one of five linear quaternion functions. Three equations give

$$c = [f_1d, f_2d, f_3d] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (127),$$

and substitution in the fourth and fifth require the point d to be on the curve of the quartic surfaces

$$(f_1d, f_2d, f_3d, f_4d) = 0, \quad (f_1d, f_2d, f_3d, f_5d) = 0 \quad . \quad . \quad . \quad . \quad (128),$$

* This condition appears to answer in every particular the condition (compare 'Elements of Quaternions,' New Ed., vol. ii., p. 377) that a triangle can be inscribed to one conic and circumscribed to another (*see*, however, SALMON'S 'Three Dimensions,' Note to Art. 207).

which is complementary to the sextic curve (compare Art. 64),

$$[f_1 d, f_2 d, f_3 d] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (129).$$

Selecting any point d on this complementary curve of the tenth order, c is determined by (127), and the sixth condition must be satisfied.

Hence it appears that any two vertics may be assumed at random, and a plane locus for the third. Ten points d lie in this plane, and ten tetrahedra satisfy the conditions.

Generally, also, if the sum of the products of the square roots of the latent roots of the function vanishes, an infinite number of tetrahedra may be found related to their correspondents, so that corresponding edges a, b; a', b', are intersected by opposite edges of intermediate tetrahedra. (Compare Art. 25.)

28. The case in which the two invariants n' and n''' vanish simultaneously is of considerable importance in the theory of the linear function. These conditions are always satisfied for the functions $2f_{\prime} = f - f'$; and also for functions of a more general type; in fact, for functions whose squares satisfy a depressed equation

$$(f^2)^2 + n''f^2 + n = 0, \text{ or } (f^2 - s^2)(f^2 - s'^2) = 0 \quad . \quad . \quad . \quad (130).$$

It appears from Art. 24 that two systems of tetrahedra exist, one set having their correspondents inscribed to them, the other set being inscribed to their correspondents. We shall prove that *one system of tetrahedra exists which are at once inscribed and circumscribed to their correspondents.*

Let q_1 and q_2 be the united points of f for the roots $\pm s$, and q'_1 and q'_2 for the roots $\pm s'$. Take any line whatever

$$q = x(q_1 + uq_2) + y(q'_1 + vq'_2) \qquad (x, y \text{ variable}) \quad . \quad . \quad . \quad . \quad (131),$$

intersecting the lines $q_1 q_2$ and $q'_1 q'_2$. The function f converts this line into the line

$$p = xs(q_1 - uq_2) + ys'(q'_1 - vq'_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (132),$$

which intersects the connectors of the united points in the harmonic conjugates of the points of intersection of the original line. Repeating the operation, the line p is restored to q.

In other words, when n' and n'' vanish, *the transformation interchanges lines which cut harmonically the connectors of the united points;* or it transforms a certain congeneney of lines into itself.

Take any tetrahedron having opposite edges, ab and cd, on two conjugate lines of this congenency; the corresponding tetrahedron has the two edges $c'd'$ and $a'b'$ respectively on those two lines, and either tetrahedron may be said to be at one and the same time inscribed and circumscribed to the other.

If the line a, b intersects the connectors in the points Q_1 and Q'_1, and if a', b' intersects them in Q_2, Q'_2 (compare (131), (132)), we may write

$$a = Q_1 + t_1 Q'_1\,; \quad b = Q_1 + t_2 Q'_1\,; \quad c' = sQ_1 + s't_3 Q'_1\,; \; d' = sQ_1 + s't_4 Q'_1\,;$$
$$a' = sQ_2 + s't_1 Q'_2\,; \; b' = sQ_2 + s't_2 Q'_2\,; \; c \; = Q_2 + t_3 Q'_2\,; \quad d = Q_2 + t_4 Q'_2\,;$$

and the anharmonics of the ranges $abc'd'$ and $a'b'cd$ are

$$\frac{(ab)(c'd')}{(bc')\,(d'a)} = \frac{ss'(t_1 - t_2)(t_3 - t_4)}{(st_2 - s't_3)(s't_4 - st_1)}\,; \; \frac{(a'b')(cd)}{(b'c)(da')} = \frac{ss'(t_1 - t_2)(t_3 - t_4)}{(s't_2 - st_3)(st_4 - s't_1)}.$$

For a pair of quadrics (118) a quadrilateral on one determines a self-conjugate tetrahedron with respect to the other if n' and n''' of the function $F_1^{-1}F_2$ vanish. Moreover, in this case the quadrics

$$SqF_1 q = 0, \quad SqF_2 F_1^{-1} F_2 q = 0$$

intersect in a common quadrilateral.

29. It may be worth while drawing attention to a simple rule for obtaining in a convenient form certain scalar invariants of linear functions. These invariants are the coefficients of powers and products of x_1, x_2, &c., in the latent quartic of the function

$$x_1 f_1 + x_2 f_2 + \ldots + x_n f_n$$

and the rule is to distinguish by accents or suffixes the symbols in $(abcd)$ just as if this expression had been differentiated. For instance, there is the twelve-term invariant

$$n_{12}\,(abcd) = \Sigma(a_1 b_2 cd)$$

where a_1 stands for $f_1 a$, and a_2 for $f_2 a$.

It would appear that when a twelve-term invariant vanishes, every term will vanish provided the tetrahedron $(abcd)$ is suitably inscribed to a definite curve.

Suppose eleven terms vanish. Let three be solved for a, and substitution in the remaining eight leaves eight equations in b, c and d. From three of these find b, and five are left in c and d; and on elimination of c, two equations in d remain, which represent a definite curve. From symmetry the remaining three vertices trace out a curve or curves. These curves are covariant with the functions.

SECTION IV.

THE RELATIONS OF A PAIR OF QUADRICS, $SqF_1q = 0$, $SqF_2q = 0$, WHICH DEPEND ON THE NATURE OF THE FUNCTION $F_2^{-1}F_1$.

30. We shall briefly consider the relations of a pair of quadrics which depend on the peculiarities of the function $F_2^{-1}F_1$, where

$$SqF_1q = 0, \quad SqF_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (133)$$

are the equations of the two quadrics.

If the polar plane of the point a is the same with respect to the two quadrics,

$$F_1a = t_1F_2a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (134),$$

where t_1 is a scalar, because (Art. 11) the symbols of the polar planes are F_1a and F_2a. Here t_1 is a latent root of the function $F_2^{-1}F_1$ and a is a united point.

If b is a second united point answering to the latent root t_2, we have, on account of the self-conjugate character of the functions F_1 and F_2,

$$t_1SbF_2a = SbF_1a = SaF_1b = t_2SaF_2b = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (135),$$

provided the latent roots are distinct. Thus the polar plane of a contains the points b, c, and d; and the tetrahedron is self-conjugate to both quadrics. The function $F_1^{-1}F_2$ belongs to the general type I_1, in which all the united points are distinct (Art. 13).

31. Let two united points a and b approach coincidence. The relation (135) remains true up to the limit, and ultimately

$$SaF_1a = 0, \quad SaF_2a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (136);$$

and the coalesced point is situated on the curve of intersection of the surface. By (134) the symbols of the tangent planes to the two surfaces are identical, and the two surfaces touch.

If then $F_2^{-1}F_1$ belongs to the type I_2, the surfaces touch, and conversely; and if the quadrics touch in two distinct points the type of the function is I_5, and the intersection is a line and a cubic.

Let c and d be the remaining united points. By (135) the line c, d lies in the common tangent plane; so in order to determine the generators of the two quadrics in the plane, it is only necessary to determine the points in which the quadrics meet the line c, d. For the first and second quadrics, the equations determining the points $c + xd$ are respectively (134)

$$t_3 \mathrm{S}c\mathrm{F}_2 c + x^2 t_4 \mathrm{S}d\mathrm{F}_2 d = 0 \quad \mathrm{S}c\mathrm{F}_2 c + x^2 \mathrm{S}d\mathrm{F}_2 d = 0 ; \quad . \quad . \quad . \quad (137).$$

The quadrics consequently have distinct generators unless $t_3 = t_4$, and unless the points c and d are distinct.

For quadrics having a pair of common co-planar generators, $\mathrm{F}_2^{-1}\mathrm{F}_1$ is of the type II_2, and conversely.

32. In the next place, let three roots t_1 be equal, so that a is the union of three united points of $f = \mathrm{F}_2^{-1}\mathrm{F}_1$. The point a' of Art. 15 (78) is now in the common tangent plane, because it has been derived by the operation of $f - t_1$ from another point a''. In fact we have

$$(\mathrm{F}_1 - t_1 \mathrm{F}_2)\, a'' = \mathrm{F}_2 a', \quad (\mathrm{F}_1 - t_1 \mathrm{F}_2)\, a' = \mathrm{F}_2 a \quad . \quad . \quad . \quad . \quad (138);$$

and from the first of these it is obvious that $\mathrm{S}a\mathrm{F}_2 a' = 0\ (= t_1^{-1}\mathrm{S}a\mathrm{F}_1 a')$, while the second may be written in the form

$$\mathrm{F}_1 (a + xa') = (t_1 + x)\, \mathrm{F}_2 \left(a + \frac{xt_1}{t_1 + x} a' \right) \quad . \quad . \quad . \quad . \quad . \quad (139).$$

This equation shows that the polar plane of the point $a + xa'$ with respect to the first quadric is identical with the polar plane of $a + \frac{xt_1}{t_1 + x} a'$ with respect to the second; and because a' lies in the tangent plane, in the limit where x becomes infinitesimally small, the two points become identical to the first order of x, and the common polar plane becomes a consecutive tangent plane to both quadrics. The quadrics have, therefore, stationary contact, and their function $\mathrm{F}_2^{-1}\mathrm{F}_1$ is of the class I_3.

The generators in the tangent plane are now found by expressing that $xa' + d$ is on one of the quadrics; the equations may be written in the form

$$x^2 t_1 \mathrm{S}a'\mathrm{F}_2 a' + 2xt_1 \mathrm{S}a'\mathrm{F}_2 d + t_4 \mathrm{S}d\mathrm{F}_2 d = 0 ; \; x^2 \mathrm{S}a'\mathrm{F}_2 a' + 2x\mathrm{S}a'\mathrm{F}_2 d + \mathrm{S}d\mathrm{F}_2 d = 0 \quad . \quad (140),$$

where the equation for the first quadric has been reduced by the aid of (138), in order that it may be compared with that for the second quadric. The generators are common if, and only if, $t_1 = t_4$, and the function is then of the type II_4.

33. When the four united points coincide, the point a'' as well as a' lies in the common tangent plane, a'' having been derived, as a' was in the last article, from a third point a'''. From the three equations

$$(F_1 - t_1F_2)\alpha''' = F_2\alpha'' ; \quad (F_1 - t_1F_2)\alpha'' = F_2\alpha' ; \quad (F_1 - t_1F_2)\alpha' = F_2\alpha \quad . \quad (141),$$

we see that, in addition to the conditions that the points should lie in the tangent plane, we have

$$S\alpha''(F_1 - t_1F_2)\alpha' = 0 ; \quad S\alpha'F_2\alpha' = 0, \quad \text{and} \quad S\alpha'F_1\alpha' = 0 \quad . \quad . \quad (142),$$

as appears from operating on the third by $S\alpha''$ and using this result in operating on the second by $S\alpha'$, and finally operating on the third by $S\alpha'$. The line $a + x\alpha'$ is consequently a generator of both quadrics, and the function belongs to the class I_4.

The remaining generators, determined by the point in which $\alpha' + y\alpha''$ meets the surfaces again, are deducible from the equations

$$t_1S\alpha'F_2\alpha'' + yt_1S\alpha''F_2\alpha'' + yS\alpha''F_2\alpha' = 0 ; \quad S\alpha'F_2\alpha'' + yS\alpha''F_2\alpha'' = 0 \quad . \quad (143).$$

If these remaining generators are common to both quadrics we must have $S\alpha''F_2\alpha' = 0$, and then they coincide of necessity with the other generator, and the quadrics become a pair of cones touching along a generator.

34. Suppose the function to have a line locus of united points, so that

$$F_1a = t_1F_2a ; \; F_1b = t_1F_2b \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (144) ;$$

it immediately follows that one quadric meets the line a, b in two points common to the other, and the quadrics touch at these two points. Substituting in the equations of the quadrics

$$q = xa + yb + z(c + ud) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (145),$$

the equations become,

$$t_1S(xa + yb)F_2(xa + yb) + z^2(t_3ScF_2c + u^2t_4SdF_2d) = 0$$
$$S(xa + yb)F_2(xu + yb) + z^2(ScF_2c + u^2SdF_2d) = 0 \quad . \quad . \quad . \quad . \quad (146),$$

and for a constant value of u these represent the sections by an arbitrary plane through the line a, b. These sections are identical if

$$(t_3 - t_1)ScF_2c + u^2(t_4 - t_1)SdF_2d = 0 \quad . \quad . \quad . \quad . \quad . \quad (147),$$

and as this is a quadratic in u, the quadrics have two plane sections common. The function f is of the type II. The case of coincidence of the points c, d has occurred in Art. 31, one of the conics breaking up (type II_2).

If $t_3 = t_1$, while c is not situated on ab, the quadrics have two coincident plane sections, or ring-contact. The type of the function is III_1.

If $t_3 = t_4$, but c not coincident with d, the function is of the class IV., and the quadrics intersect in common points on the line c, d. Let ab meet the quadrics in a', b' and cd in $c'd'$, then it is very easy to see that a', c', b', d' is a quadrilateral common to both surfaces.

When c coincides with a point a on the line, let a' be the point for which (Art. 15)

$$(\mathrm{F}_1 - t_1\mathrm{F}_2)\, a' = \mathrm{F}_2 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (148),$$

then $\mathrm{S}a\mathrm{F}_2 a = 0$, and $\mathrm{S}b\mathrm{F}_2 a = 0$, and the line ab touches the two quadrics at a. The conics in the common plane sections touch (type II_3).

If, further, d coincides with the point a (type II_4), the point $\mathrm{F}_2 a'$ is derived by the operation of $\mathrm{F}_1 - t_1\mathrm{F}_2$ from some other point a'' (Art. 32), and therefore

$$\mathrm{S}a\mathrm{F}_2 a' = 0\,;\ \ \mathrm{S}b\mathrm{F}_2 a' = 0\,;\ \text{ and }\ \mathrm{S}a\mathrm{F}_1 a' = 0\,;\ \ \mathrm{S}b\mathrm{F}_1 a' = 0 \quad . \quad . \quad . \quad (149).$$

Hence it appears that the line $a' + xb$ meets the two quadrics in the same two points, and the lines from a to these points are common generators. The intersection of the quadrics consists, therefore, of a pair of lines and a conic passing through their common point (type II_4).

Finally, it remains to notice the case of a plane locus of united points with the fourth point in the plane (III_2). It may be proved that in this case the coincident plane sections consist of a pair of lines along which the quadrics touch.

35. Summing up, the intersection of two quadrics according to the types of the function $\mathrm{F}_2^{-1}\mathrm{F}_1$, is

I_1, a twisted quartic with two apparent double points ;
I_2, a twisted quartic with three apparent double points ;
I_3, a twisted quartic with two apparent and one real double point ;
I_4, a right line and a cubic touching it ;
I_5, a right line and a cubic ;
II_1, two conics ;
II_2, a pair of lines and a conic ;
II_3, two conics in contact ;
II_4, a pair of lines and a conic through their intersection ;
III_1, the surfaces touch along a conic ;
III_2, the surfaces touch along two generators ;
IV, the intersection is a quadrilateral.

SECTION V.

The Square Root of a Linear Quaternion Function.

36. When the same effect is produced by the twice-repeated operation of one linear quaternion function and by the single operation of another, the former may be said to be a square root of the latter.

TAIT first extracted the square root of a linear vector function, and pointed out the great utility of the conception. We now proceed to examine some of the properties of a square root of a quaternion function, and to illustrate their bearing on certain geometrical investigations.

The united points of a square root are also united points of the primitive function.

If

$$f^{\frac{1}{2}}a = t_1^{\frac{1}{2}}a, \quad \text{then} \quad fa = t_1 a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (150).$$

The converse does not hold, for it may happen that loci of united points exist for the primitive and not for the square root. For example, if

$$f^{\frac{1}{2}}a = b, \quad f^{\frac{1}{2}}b = t_1 a\,; \quad \text{then} \quad fa = t_1 a, \quad fb = t_1 b \quad . \quad . \quad . \quad . \quad (151),$$

and though every point on the line $a\,b$ is a united point for the primitive, this is not generally true for its square roots. (Compare Art. 13.)

When there is no locus of united points, the square roots have the same four united points as the primitive, and their latent roots are sets of the square roots

$$\pm t_1^{\frac{1}{2}}, \quad \pm t_2^{\frac{1}{2}}, \quad \pm t_3^{\frac{1}{2}}, \quad \pm t_4^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (152)$$

of the latent roots of the primitive. Thus in the general case a function has sixteen square roots.

37. *When the primitive has a line locus of united points* (a, b), *any two points on the line may be assumed as united points of the square root.*

By the last article it may be seen that the square root must have united points on the line. Assume these to be $a + xb$, $a + yb$, then

$$f^{\frac{1}{2}}(a + xb) = \pm\, t_1^{\frac{1}{2}}(a + xb)\,; \quad f^{\frac{1}{2}}(a + yb) = \pm\, t_1^{\frac{1}{2}}(a + yb) \quad . \quad . \quad (153),$$

and the square root satisfies the condition that its twice repeated operation is equivalent to the operation of f. If the signs are alike and x and y distinct, the square root has a locus of united points; otherwise it has not.

If a square root has coalesced points, so has the primitive.

If

$$f^{\frac{1}{2}}a' = t^{\frac{1}{2}}a' + a\,; \quad f^{\frac{1}{2}}a' = t^{\frac{1}{2}}a\,; \quad \text{then}\, fa' = ta' + 2t^{\frac{1}{2}}a\,; \quad fa = ta. \quad . \quad (154),$$

and therefore the repeated operation of $f - t$ is required to destroy a'; and the primitive has a coalesced united point.

The square root of a function having a plane locus of united points possesses at least a line locus of united points.

The only escape is the assumption that the square root has a united point coalesced from three points, and this has just been shown to involve a coalesced point for the primitive, contrary to hypothesis.

When the primitive has coalesced points but no loci of united points, the number of square roots is limited.

This follows from (154).

38. *Except in the case in which the primitive has loci of united points, the square roots are all commutative with one another and with the primitive,* for they possess a common system of united points.*

Moreover, for a definite square root,

$$(f+x)^{\frac{1}{2}}(f+y)^{\frac{1}{2}} = ((f+x)(f+y))^{\frac{1}{2}} \quad \ldots \ldots \quad (155),$$

with liberty to change the order of the factors. This follows most easily by operating on the united points.

In general also, for any two functions f_1 and f_2, and a definite square root,

$$f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} = (f_1^{\frac{1}{2}} f_2^2 f_1^{-\frac{1}{2}})^{\frac{1}{2}} \quad \ldots \ldots \quad (156),$$

because

$$(f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}})^2 = f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} \cdot f_1^{\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} = f_1^{\frac{1}{2}} f_2^2 f_1^{-\frac{1}{2}} \quad \ldots \ldots \quad (157);$$

and in particular a relation which is occasionally useful is

$$(f_1^{-1} f_2 + t)^{\frac{1}{2}} = f_1^{-\frac{1}{2}} (f_1^{-\frac{1}{2}} f_2 f_1^{-\frac{1}{2}} + t)^{\frac{1}{2}} f_1^{\frac{1}{2}} \quad \ldots \ldots \quad (158).$$

39. It is evident from the foregoing that the square roots of a function and of its conjugate are conjugate when they have the same latent roots.

Thus we may write

$$(f^{\frac{1}{2}})' = f'^{\frac{1}{2}} \quad \ldots \ldots \quad (159),$$

to signify that the conjugate of a square root is the corresponding square root of the conjugate function.

In particular, taking the conjugate of (158),

$$(f_2' f_1'^{-1} + t)^{\frac{1}{2}} = f_1'^{\frac{1}{2}} (f_1'^{-\frac{1}{2}} f_2' f_1'^{-\frac{1}{2}} + t)^{\frac{1}{2}} f_1'^{-\frac{1}{2}} \quad \ldots \ldots \quad (160).$$

SECTION VI.

The Square Root of a Function in relation to the Geometry of Quadrics.

40. The transformation

$$p = f^{\frac{1}{2}} q \quad \ldots \ldots \quad (161)$$

converts the quadric $Sqfq = 0$ into the unit sphere $Sp^2 = 0$, f being a self-conjugate function.

* Compare 'Elements of Quaternions,' New Ed., vol. ii., Appendix, p. 364.

This suggests a quaternion equation such as

$$q = (f+x)^{\frac{1}{2}}(f+y)^{\frac{1}{2}}(f+z)^{\frac{1}{2}}e = \sqrt{\{(f+x)(f+y)(f+z)\}}\,e \quad . \quad (162),$$

where e is some constant quaternion, as equivalent to the equation of a system of generalized confocals

$$Sq(f+x)^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (163).$$

On substitution in the scalar from the quaternion equation the result is

$$Se(f+y)(f+z)e = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (164),$$

and y and z disappear, provided e is chosen to be one of the eight points satisfying

$$Se^2 = Sefe = Sef^2e = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (165).$$

Thus e is one of the intersections of three known quadrics.

It is not necessary to dwell on HAMILTON's theory of the umbilicar generatrices, as the subject will be resumed in an extended form.* Accordingly it is sufficient to mark that the equation of such a generator is

$$q = (f+y)(f+x)^{\frac{1}{2}}e = (f+x)^{\frac{3}{2}}e + \tfrac{2}{3}(y-x)\frac{d}{dx}(f+x)^{\frac{3}{2}}e \quad . \quad . \quad (166),$$

where y is variable; and the form of this equation shows that when x varies the generator sweeps out the developable of which the cuspidal edge is the curve

$$q = (f+x)^{\frac{3}{2}}e \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (167).$$

41. More generally, starting from any two quadrics,

$$Sqf_1q = 0, \quad Sqf_2q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (168);$$

the equation of the system of quadrics inscribed to their common circumscribing developable (compare Art. 11) is

$$Sq(f_1^{-1} + xf_2^{-1})^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (169).$$

This by the principles of Art. 38 may be replaced by

$$Sqf_2^{\frac{1}{2}}(f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + x)^{-1}f_2^{\frac{1}{2}}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (170);$$

and on comparison with (163) and (162) it is manifestly equivalent to the quaternion equation

$$f_2^{\frac{1}{2}}q = (f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + x)^{\frac{1}{2}}(f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + y)^{\frac{1}{2}}(f_2^{\frac{1}{2}}f_1^{-1}f_2^{\frac{1}{2}} + z)^{\frac{1}{2}}e' \quad . \quad . \quad . \quad (171);$$

or, by an application of (158), to

$$q = (f_1^{-1}f_2 + x)^{\frac{1}{2}}(f_1^{-1}f_2 + y)^{\frac{1}{2}}(f_1^{-1}f_2 + z)^{\frac{1}{2}}e \quad . \quad . \quad . \quad . \quad (172),$$

* Compare Arts. 41 and 71.

where $e = f_2^{-\frac{1}{2}}e'$. By (165) it is seen that the quaternion e of this formula satisfies the three equations

$$Sef_2e = 0, \quad Sef_2f_1^{-1}f_2e = 0, \quad Sef_2f_1^{-1}f_2f_1^{-1}f_2e = 0 \quad . \quad . \quad . \quad (173),$$

and is therefore one of the intersections of three quadrics.

42. In particular the equation of the curve of intersection of the original quadrics (168) is

$$q = (f_1^{-1}f_2 + x)^{\frac{1}{2}}a, \text{ where } Saf_1a = Saf_2a = Saf_1f_1^{-1}f_2a = 0 \quad . \quad . \quad (174),$$

as may be proved by direct transformation from the general result (172), or perhaps more shortly by assuming the form $q = (f + x)^{\frac{1}{2}}a$ and determining f, or by verification. remembering (158).

Hence the equation

$$Sa\,(f_2f_1^{-1} + x)^{\frac{1}{2}}f_3(f_1^{-1}f_2 + x)^{\frac{1}{2}}a = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (175)$$

determines the eight points of intersection of the three quadrics

$$Sqf_1q = 0, \quad Sqf_2q = 0, \quad Sqf_3q = 0.$$

SECTION VII.

The Family of Curves $q = (f + t)^m a$ and their Developables.

43. Instead of writing down and discussing the equations of the circumscribing developable and of its cuspidal edge of the quadrics (169), which are in fact of the same form as (166) and (167), except that $f = f_1^{-1}f_2$ is not self-conjugate, we shall devote a few remarks to the family of curves

$$q = (f + t)^m a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (176)$$

and their developables, m being a scalar, a a constant quaternion, t a scalar variable, and f an arbitrary linear quaternion function. This family includes the right line, the conic, the twisted cubic, the quartic intersection of two quadrics, the quartic which is not the intersection of two quadrics, and the cuspidal edge of the developable circumscribed to two quadrics; the corresponding values of m being $m = 1, 2, -1$ or 3, 4 and $\frac{3}{2}$.

44. The equation of a tangent to the curve (176) is

$$q = (f + s)\,(f + t)^{m-1}a \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (177),$$

when the scalar parameter s alone varies. When s and t both vary the equation is that of the developable of the tangent lines.

If for suitable weights of the united points q_1, q_2, q_3, q_4, we write

$$a = q_1 + q_2 + q_3 + q_4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (178),$$

the equation of the developable becomes

$$q = \sum_{1}^{4} (t_1 + s)(t_1 + t)^{m-1} q_1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (179).$$

When $m - 1$ is positive, the result of putting $t = - t_1$ is

$$q = (t_2 + s)(t_2 - t_1)^{m-1} q_2 + (t_3 + s)(t_3 - t_1)^{m-1} q_3 + (t_4 + s)(t_4 - t_1)^{m-1} q_4 \,. \quad . \,(180);$$

and this represents a certain number of right lines in the united plane $[q_2, q_3, q_4]$, the number being determined by the nature of m, being as we know 4 when the developable is circumscribed to a pair of quadrics, or when $m = \frac{3}{2}$.

The remaining part of the intersection in the united plane is obtained by putting s equal to $-t_1$, and its equation is

$$q = (t_2 - t_1)(t_2 + t)^{m-1} q_2 + (t_3 - t_1)(t_3 + t)^{m-1} q_3 + (t_4 - t_1)(t_4 + t)^{m-1} q_4 \,. \quad . \,(181);$$

or more simply

$$q = (f + t)^{m-1} a_1, \quad \text{where} \quad a_1 = (t_2 - t_1) q_2 + (t_3 - t_1) q_3 + (t_4 - t_1) q_4 \,. \quad . \,(182).$$

The plane curve is likewise included in the family (176), and for $m = \frac{3}{2}$ it is a quartic (174), or rather a conic counted twice,

$$x_2^2 \frac{t_3 - t_4}{(t_2 - t_1)^2} + x_3^2 \frac{t_4 - t_2}{(t_3 - t_1)^2} + x_4^2 \frac{t_2 - t_3}{(t_4 - t_1)^2} = 0 \quad . \quad . \quad . \quad . \quad (183),$$

as we see from (181) on putting $q = x_2 q_2 + x_3 q_3 + x_4 q_4$.

In case $m - 1$ is negative it is necessary first to multiply (179) by the product $\Pi (t_1 + t)^{1-m}$ before putting $- t$ equal to a latent root. Then, on making $t = - t_2$, we find only the point q_2, which shows there are no right lines in the plane $[q_2 q_3 q_4]$, and which indicates multiplicity of the curve at the united points.

45. Just as the equation of the tangent line was obtained in the last article from that of the curve, the equation of the osculating plane may be written in the form

$$q = (f + u)(f + s)(f + t)^{m-2} a \quad . \quad . \quad . \quad . \quad . \quad . \quad (184);$$

where t is supposed to remain constant, while s and u vary together. It is easy to verify that this plane contains two consecutive tangents to the curve.

The reciprocal of the plane is the point (compare Art. 5)

$$p = (f' + t)^{2-m} a', \qquad a' = [a, fa, f^2 a] \quad . \quad . \quad . \quad . \quad . \quad (185);$$

and consequently the cuspidal edges of the reciprocals of curves of the family (176) belong to a similar family obtained by altering a into a' and f into its conjugate. Also the sum of the exponents m for a curve and the cuspidal edge of its reciprocal is equal to 2.

The developable formed by the tangents to the new cuspidal edge is

$$p = (f' + s')(f' + t)^{1-m}a' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (186);$$

and it may be worth while to verify directly that lines of this reciprocal developable are reciprocal to the corresponding lines of (179). Also lines in a united plane reciprocate into lines through a united point of the conjugate function; so that we can assert that the number of lines of the developable of a curve whose exponent is m which lie in a united plane is the number of lines of the developable of a curve whose exponent is $2 - m$ which pass through a united point.

46. The points (s) in which an osculating plane (184) at (t) cuts the curve again are found by combining this equation with (176) and putting

$$Sqp = 0 = S(f + s)^m a (f' + t)^{2-m} a' = Sa'(f + t)^{2-m} (f + s)^m a \quad . \quad . \quad (187).$$

In this, when we use the expression (178) for a and when we observe (185) that

$$a' = [afaf^2a] = \Sigma[q_2 q_3 q_4](t_2 - t_3)(t_3 - t_4)(t_4 - t_2) \quad . \quad . \quad . \quad . \quad (188),$$

equation (187) becomes

$$\Sigma \frac{(t_1 + s)^m}{(t_1 + t)^m}(t_1 + t)^2(t_2 - t_3)(t_3 - t_4)(t_4 - t_2) = 0 \quad . \quad . \quad . \quad . \quad (189).$$

The points at which the plane meets the curve four times are determined by

$$\Sigma (1)^m (t_1 + t)^2 (t_2 - t_3)(t_3 - t_4)(t_4 - t_2) = 0 \quad . \quad . \quad . \quad . \quad . \quad (190).$$

SECTION VIII.

The Dissection of a Linear Function.

47. In addition to the decomposition of a function into its self-conjugate and non-conjugate parts by addition and subtraction, there is another very useful resolution by multiplication and division analogous to Tait's resolution of a linear

vector function into a function representing a pure strain following or followed by a rotation.

Multiply any function into its conjugate, and write

$$ff' = F^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (191),$$

where F is the self-conjugate function whose double operation is equivalent to the operation of the self-conjugate function ff' (Art. 36).

Introducing a new linear function g and its conjugate g' defined by the relations

$$f = Fg, \quad f' = g'F \quad \text{or} \quad g = F^{-1}f, \; g' = f'F^{-1} \quad . \quad . \quad . \quad . \quad (192),$$

it appears that this function is the inverse of its conjugate, for

$$g'g = 1 = gg' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (193)$$

is a consequence of the equations of definition.

The geometrical property of this new function is, that *points conjugate to the unit sphere remain conjugate after transformation.*

For if

$$Spq = 0, \quad \text{then} \quad Sgpgq = Spg'gq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (194).$$

In particular the unit sphere is converted into itself by the transformation.

This transformation is orthogonal, points and planes being transformed by the same function (Art. 4).

48. On counting the constants, it appears that an arbitrary function f cannot be reduced to the product of a self-conjugate function and a conical rotator

$$R = r(\;)r^{-1}, \quad R' = R^{-1} = r^{-1}(\;)r \quad . \quad . \quad . \quad . \quad . \quad . \quad (195),$$

there being sixteen constants in f, ten in F, and three in R.

In order to determine the conditions, observe that by the last article

$$F^2 = ff' \quad \text{if} \quad f = FR, \quad \text{and} \quad RR' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (196).$$

Now I say that *if a scalar remains a scalar after the operation of* R, *the function is a conical rotator.* For then

$$SR'\rho = S\rho R(1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (197),$$

and therefore $R'\rho$ or $R\rho$ remains a vector whatever vector ρ may be; and, moreover, the angle between any two vectors is unaltered by the transformation.*

Thus the condition required is simply

$$f(1) = F(1), \quad \text{where} \quad F^2 = ff' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (198);$$

and when the reduction is possible it is generally determinate.

* Compare the Appendix to the New Edition of HAMILTON'S 'Elements,' vol. ii., p. 366.

49. *A function which is the inverse of its conjugate is in general reducible in an infinite variety of ways to the product of a self-conjugate function and a rotator.*

Because $gg' = 1$ in the notation of Art. 47, the conditions (198) that g should be reducible are

$$g(1) = \mathrm{G}(1), \quad \text{where} \quad \mathrm{G}^2 = 1,\ \mathrm{G} = \mathrm{G}' \quad \text{(199)},$$

for simplicity writing

$$1 + g(1) = a, \quad 1 - g(1) = b \quad \text{(200)};$$

it is evident from the last equation that

$$\mathrm{G}a = a, \quad \mathrm{G}b = -b, \quad \mathrm{S}ab = 0 \quad \text{(201)};$$

so a and b are united points of G, and conjugate with respect to the unit sphere.

Take any point c in the polar plane of b, and any point d in the polar line of ac; and assume

$$\mathrm{G}c = c, \quad \mathrm{G}d = -d \quad \text{(202)};$$

then the function determined by the four relations (201) and (202) is self-conjugate, and its symbolic equation is $\mathrm{G}^2 - 1 = 0$. By the construction it follows that

$$\mathrm{S}ab = \mathrm{S}bc = \mathrm{S}ad = \mathrm{S}cd = 0 \quad \text{(203)},$$

and the function is consequently self-conjugate.

We have now determined a self-conjugate function, one of an infinite number, which satisfies (199), and the proposition is proved.

The rotator corresponding to G is of course

$$\mathrm{R} = \mathrm{G}^{-1}g = \mathrm{G}g \quad \text{(204)}.$$

50. The results of recent articles establish the possibility of reducing an arbitrary function to the form

$$f = \mathrm{FGR} \quad \text{(205)};$$

where F, G, and R satisfy the equations

$$\mathrm{F}^2 = ff', \quad \mathrm{F}^{-1}f(1) = \mathrm{G}(1), \quad \mathrm{G}^2 = 1, \quad \mathrm{R} = \mathrm{GF}^{-1}f \quad \text{(206)};$$

and by analogous processes the function may also be reduced to other forms such as $\mathrm{G}_1\mathrm{F}_1\mathrm{R}_1$, but on these we need not delay.

51. *An arbitrary function may be reduced to a quotient or product of two self-conjugate functions.*

Assuming

$$f = \mathrm{F}_2^{-1}\mathrm{F}_1 \quad \text{(207)},$$

it appears that the united points of f (compare Art. 30) satisfy the equations

$$\mathrm{F}_1a = t_1\mathrm{F}_2a; \quad \mathrm{F}_1b = t_2\mathrm{F}_2b; \quad \mathrm{F}_1c = t_3\mathrm{F}_2c; \quad \mathrm{F}_1d = t_4\mathrm{F}_2d \quad \text{(208)};$$

but on the supposition that F_1 and F_2 are self-conjugate, it follows (135) that these united points form a tetrahedron self-conjugate to the two quadrics $SqF_1q = 0$, $SqF_2q = 0$. Take therefore any quadric to which this tetrahedron is self-conjugate; F_1 is determined and F_2 follows from (207).

Otherwise the function

$$F_2(q) . (abcd) = xa'(qbcd) + yb'(aqcd) + zc'(abqd) + wd'(abcq) \quad . \quad (209)$$

is self-conjugate (Art. 21) when $(a'b'c'd')$ is the tetrahedron reciprocal to $(abcd)$; and on comparison with (208) the function F_1 may be written down. The four scalars x, y, z, w are arbitrary, as might have been expected, since each self-conjugate function involves ten constants, while f involves sixteen.

If two functions can be simultaneously reduced to the forms

$$f_1 = F^{-1}F_1, \quad f_2 = F^{-1}F_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (210),$$

the united points of f_1 and f_2 must form tetrahedra self-conjugate to a common quadric, or

$$Fa_1 = x_1[b_1c_1d_1], \text{ \&c.} \quad Fa_2 = x_2[b_2c_2d_2], \text{ \&c.} \quad . \quad . \quad . \quad . \quad (211).$$

In this case the eight united points are so related that any quadric

$$SqF_3q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (212)$$

which passes through seven, passes also through the eighth.

The condition that the point a_1 should be on the quadric may be written (211)

$$Sa_1F_3F^{-1}[b_1c_1d_1] = 0, \quad \text{or} \quad (F^{-1}F_3a_1, b_1, c_1, d_1) = 0 \quad . \quad . \quad . \quad (213),$$

and if b_1, c_1, and d_1 are likewise on the quadric, it follows (Art. 24) that the first invariant of the function $F^{-1}F_3$ vanishes. Hence if the points a_2, b_2, c_2 are also on the quadric, the remaining point d_2 must lie on the quadric too.* Thus one of the united points is fixed with respect to the others, and the functions f_1 and f_2 must satisfy three conditions, which reduce the number of their constants to 29, and this is precisely the number involved in the two quotients $F^{-1}F_1$, $F^{-1}F_2$.

* Compare Appendix to the New Edition of the 'Elements of Quaternions,' vol. ii., p. 364.

SECTION IX.

The Determination of Linear Transformations which satisfy Certain Conditions.

52. The results of Art. 47 afford a simple solution of such problems as to find a *transformation which shall convert one quadric into another.*

Symbolically this problem amounts to solving the equation

$$\mathrm{F}_1 = f'\mathrm{F}_2 f \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (214)$$

which connects two known self-conjugate functions F_1 and F_2 with an unknown function f and its conjugate.

The first quadric is reduced to the unit sphere by the transformation

$$q_1 = \mathrm{F}_1^{\frac{1}{2}} q, \quad \text{so that } \mathrm{S}q\mathrm{F}_1 q = \mathrm{S}q_1^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (215).$$

The unit sphere is converted into itself by the transformation (Art. 47)

$$q_2 = g q_1, \quad \text{so that } \mathrm{S}q_1^2 = \mathrm{S}q_2^2 \text{ if } gg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (216);$$

and finally the sphere is converted into the second quadric by the transformation

$$q_3 = \mathrm{F}_2^{-\frac{1}{2}} q_2, \quad \text{so that } \mathrm{S}q_2^2 = \mathrm{S}q_3\mathrm{F}_2 q_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (217).$$

Thus the transformation

$$f = \mathrm{F}_2^{-\frac{1}{2}} g \mathrm{F}_1^{\frac{1}{2}}, \quad gg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (218)$$

converts the first quadric into the second; and evidently this is the most general transformation fulfilling the conditions.

53. *To convert an arbitrary function "f" into itself,* observe that the transformation must belong to the group (compare (218)),

$$\mathrm{F} = f_0^{-\frac{1}{2}} g f_0^{\frac{1}{2}}; \quad \text{where } f = f_0 + f_{\prime}, \ gg' = 1 \quad . \quad . \quad . \quad . \quad . \quad . \quad (219),$$

which converts the self-conjugate part f_0 of the function into itself.

The problem therefore reduces to the determination of g from the equation (compare (214))

$$f_{,} = f_0^{\frac{1}{2}} g' f_0^{-\frac{1}{2}} . f_{,} . f_0^{-\frac{1}{2}} g f_0^{\frac{1}{2}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (220).$$

The form of this equation suggests the new function

$$f_{,,} = f_0^{-\frac{1}{2}} f_{,} f_0^{\frac{1}{2}}, \quad f_{,,} + f'_{,,} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (221);$$

and the equation (220) reduces to

$$f_{,,} = g' f_{,,} g \text{ or } g f_{,,} = f_{,,} g \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (222);$$

and the problem reduces to the determination of a function g commutative with the known function $f_{,,}$.

The function g must possess the same* united points as $f_{,,}$; or g must be of the form (compare (221))

$$g = x + y f_{,,} + z f_{,,}^2 + w f_{,,}^3 ; \quad g' = x - y f_{,,} + z f_{,,}^2 - w f_{,,}^3 \quad . \quad . \quad (223).$$

Actually multiplying these expressions we find (219)

$$g g' = 1 = (x + z f_{,,}^2)^2 - (y f_{,,} + w f_{,,}^3)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad (224);$$

and as this equation must be equivalent to the latent quartic of the function $f_{,,}$, it must vanish when for $f_{,,}$ are substituted its latent roots. Now (Art. 23) the latent roots of $f_{,,}$ are identical with those of $f_{,}$, and the latent roots of the latter function (Art. 12) are of the form $\pm \sqrt{s}, \pm \sqrt{-s'}$. Substituting and reducing, we find in terms of the two invariants $n_{,}''$ and $n_{,}$ of $f_{,}$, two equations

$$\begin{aligned} 1 &= x^2 + n_{,} (2yw - z^2 - n_{,}'' w^2), \\ 0 &= 2xz - y^2 + n_{,}'' (2yw - z^2 - n_{,}'' w^2) + n_{,} w^2 \quad . \quad . \quad . \quad . \quad (225) \end{aligned}$$

connecting the four scalars x, y, z and w. Hence, reverting to the original functions, the transformation

$$\mathrm{F} = x + y f_0^{-1} f_{,} f_0 + z f_0^{-1} f_{,}^2 f_0 + w f_0^{-1} f_{,}^3 f_0 \quad . \quad . \quad . \quad . \quad . \quad (226)$$

converts the function f into itself; in other words, it converts the quadric and the linear complex

$$\mathrm{S} q f_0 q = 0, \quad \mathrm{S}' p f_{,} q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (227)$$

into themselves.

54. Passing on to the general case, let us consider the relations which must be satisfied when one function f can be converted into another F; or the conditions that a quadric and a complex can be simultaneously converted into another given quadric and another given complex.

* Compare Art. 38, and the Appendix to HAMILTON'S 'Elements,' vol. ii., p. 364.

The first quadric is converted into the second by the transformation (218) $F_0^{-\frac{1}{2}}gf_0^{\frac{1}{2}}$, and this converts the first complex into

$$Sp \,.\, F_0^{\frac{1}{2}}gf_0^{-\frac{1}{2}} \,.\, f_{\prime} \,.\, f_0^{-\frac{1}{2}}g'F_0^{\frac{1}{2}}q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (228);$$

and, on comparison with the second complex, it appear that we must have

$$F_0^{-\frac{1}{2}}F_{\prime}F_0^{-\frac{1}{2}} = gf_0^{-\frac{1}{2}}f_{\prime}f_0^{-\frac{1}{2}}g', \text{ or } F_{\prime\prime} = gf_{\prime\prime}g^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (229),$$

where we have introduced two new symbols for greater convenience.

Equation (229) requires the functions $F_{\prime\prime}$ and $f_{\prime\prime}$ to have the same latent roots (Art. 23); or again, $F_0^{-1}F_{\prime}$ and $f_0^{-1}f_{\prime}$ must have the same latent roots, and this is the sufficient condition, for it appears, on substituting a united point of $F_{\prime\prime}$ in (229), that the function g' must convert the united points of $F_{\prime\prime}$ into those of $f_{\prime\prime}$; and it is always possible to find a function g' capable of doing this, because (Art. 12) the united points of the two functions are quadrilaterals upon the unit sphere, and a function g' always converts this sphere into itself.

Thus, given two quadrics and two linear complexes, *it is possible to transform simultaneously one quadric into the other and one complex into the other whenever the latent roots of the functions* $f_0^{-}f_{\prime}$ and $F_0^{-1}F_{\prime}$ *are proportional.*

55. *To find a transformation which shall convert one conic into another.*

The essentials of the problem are contained in the equation

$$f(at^2 + 2bt + c) = w\,(a's^2 + 2b's + c') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (230).$$

In order that the right-hand number may be a quadratic function of t, it is necessary to have

$$w = (vt + v')^2, \quad s = \frac{ut + u'}{vt + v'} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (231);$$

so that on equating powers of t we obtain, in the usual notation for binary quantics,

$$fa = (a'b'c'\between uv)^2;\; fb = (a'b'c'\between uv\between u'v');\; fc = (a'b'c'\between u'v')^2 \quad . \quad . \quad . \quad . \quad (232).$$

These relations are not sufficient to determine the function; we may arbitrarily assume two quaternions d and d' and write $fd = d'$ (Art. 3). The function thus determined involves eleven arbitrary constants, the four u, u', v, v' which regulate the correspondence of point to point on the conics, and the seven (eight less one) involved in the two quaternions d and d', for multiplying these by a common factor is without effect.

56. *In order to transform simultaneously two given conics into two other conics, a single relation must exist connecting the conics.*

Affixing numeral suffixes, 1, 2, to the various symbols in (232), we obtain the system of six equations which the function f must satisfy. Any six quaternions are connected by two relations, and the equations

$$(s_1x_{11} + s_2x_{12})\,a_1 + 2\,(s_1y_{11} + s_2y_{12})\,b_1 + (s_1z_{11} + s_2z_{12})\,c_1$$
$$+ (s_1x_{21} + s_2x_{22})a_2 + 2\,(s_1y_{21} + s_2y_{22})\,b_2 + (s_1z_{21} + s_2z_{22})\,c_2 = 0 \quad . \quad (233)$$

$$(s'_1x'_{11} + s'_2\ x'_{12})\,a'_1 + 2\,(s'_1y'_{11} + s'_2y'_{12})\,b'_1 + (s'_1z'_{11} + s'_2z'_{12})c'_1$$
$$+ (s'_1x'_{21} + s'_2\,x'_{22})a'_2 + 2\,(s'_1y'_{21} + s'_2y'_{22})\,b'_2 + (s'_1z'_{21} + s'_1z'_{22})\,c'_2 = 0\,;$$

in which s_1, s_2, s'_1, s'_2 are arbitrary, but the other scalars given may be taken as determining the two pairs of relations connecting the two sets of six quaternions.

When the left-hand members of the equations analogous to (232) are multiplied by $s_1x_{11} + s_2x_{12}$, &c., and added, the sum is zero; and the sum of the right-hand members is (with an obvious abbreviation)

$$\{(s_1x_{11}+s_2x_{12},\ s_1y_{11}+s_2y_{12},\ s_1z_{11}+s_2z_{12})(u_1v_1)^2a'_1+2)(u_1v_1)(u'_1v'_1)b'_1+)(u'_1v'_1)^2c'_1\}$$
$$+ \{(s_1x_{21} + s_2x_{22},\ s_1y_{21} + s_2y_{22},\ s_1z_{21} + s_2z_{22})(u_2v_2)^2a'_2 + 2)(u_2v_2)(u'_2v'_2)\,b'_2 +$$
$$)(u'_2v'_2)^2c'_2\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (234),$$

or, for simplicity,

$$(s_1\mathrm{X}_{11} + s_2\mathrm{X}_{12})\,a'_1 + 2\,(s_1\mathrm{Y}_{11} + s_2\mathrm{Y}_{12})\,b'_1 + (s_1\mathrm{Z}_{11} + s'_2\mathrm{Z}_{12})\,c'_1$$
$$+ (s_1\mathrm{X}_{21} + s_2\mathrm{X}_{22})\,a'_2 + 2\,(s_1\mathrm{Y}_{21} + s_2\mathrm{Y}_{22})\,b'_2 + (s_1\,\mathrm{Z}_{21} + s_2\mathrm{Z}_{22})\,c'_2 = 0\,. \quad . \quad (235)$$

where X_{11} is a quadratic in $u_1\ v_1$, and $\mathrm{Y}_{11}\ \mathrm{Z}_{11}$ its successive polars to $u'_1v'_1$. This relation connecting the six quaternions must be equivalent to the second equation (233), so we may equate corresponding coefficients of quaternions, when we shall obtain six equations linear in s_1, s_2, s'_1, s'_2. Let s'_1 and s'_2 be eliminated from them. The result is the system of determinants

$$\left\|\begin{matrix} \mathrm{X}_{11}s_1+\mathrm{X}_{12}s_2 & \mathrm{Y}_{11}s_1+\mathrm{Y}_{12}s_2 & \mathrm{Z}_{11}s_1+\mathrm{Z}_{12}s_2 & \mathrm{X}_{21}s_1+\mathrm{X}_{22}s_2 & \mathrm{Y}_{21}s_1+\mathrm{Y}_{22}s_2 & \mathrm{Z}_{21}s_1+\mathrm{Z}_{22}s_2 \\ x'_{11} & y'_{11} & z'_{11} & x'_{21} & y'_{21} & z'_{21} \\ x'_{12} & y'_{12} & z'_{12} & x'_{22} & y'_{22} & z'_{22} \end{matrix}\right\| = 0. \quad (236),$$

which is equivalent to four equations. But s_1 and s_2 are arbitrary; consequently this system of determinants breaks up into two independent systems, equivalent to eight equations among the eight scalars u, v. The eight scalars enter homogeneously into the equations, and may be eliminated, leaving a single condition connecting the four conics, in order that it may be possible to find a transformation which shall convert two of them into the remaining two.

57. *A twisted cubic may be transformed into another twisted cubic with arbitrary correspondence of the points.*

The equation of transformation of one arbitrary twisted cubic into another is (compare Art. 43)

$$f(abcd)(t,\ 1)^3 = (a'b'c'd')(ut + u',\ vt + v')^3 \quad . \quad . \quad . \quad . \quad . \quad (237).$$

Hence equating coefficients of t, four equations are obtained which serve to deter-

mine f for arbitrary values of u, v, u', v' (Art. 3). These four scalars may be selected in any way we please.

58. *A single condition connects two quartics of the second class* when it is possible to transform one into the other.*

The equation of transformation is

$$f(abcde\between t,\ 1)^4 = (a'b'c'd'e'\between ut + u',\ vt + v')^4 \quad . \quad . \quad . \quad . \quad . \quad (238),$$

and five equations of condition may be written down analogous to (232).

Let the relations connecting the sets of five quaternions be

$$x_0 a + 4x_1 b + 6x_2 c + 4x_3 d + x_4 e = 0,\ y_0 a' + 4y_1 b' + 6y_2 c' + 4y_3 d' + y_4 e' = 0 \ . \ (239);$$

then, as in Art. 56 (234), we obtain the equation

$$X_0 a' + 4X_1 b' + 6X_2 c' + 4X_3 d' + X_4 e' = 0 \ . \quad . \quad . \quad . \quad . \quad (240),$$

where

$$X_0 = (x_0 x_1 x_2 x_3 x_4 \between uv)^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (241),$$

and where X_1, X_2, X_3, and X_4 are its successive polars to $u'v'$.

On comparison of (240) and (239) the equality of ratios

$$\frac{X_0}{y_0} = \frac{X_1}{y_1} = \frac{X_2}{y_2} = \frac{X_3}{y_3} = \frac{X_4}{y_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (242)$$

is seen to be necessary. This is equivalent to four quartic equations in the homogeneous variables u, v, u', v', and the resultant of these four equations equated to zero is the single condition in question.

SECTION X.

Covariance of Functions.

59. The subject of covariance naturally arises in connection with the various transformations lately considered, but as the principles laid down in the note on Invariants of Linear Vector Functions printed in the Appendix to the new edition of 'Hamilton's Elements' apply with but slight modification to the more general case of quaternion functions, it does not seem desirable to go into any great detail.

* A quartic of the second class is the partial intersection of a cubic and quadric surface, and only one quadric can be drawn through it.

We propose to obtain functions from given functions f_1, f_2, f_3, &c., which fall into certain classes connected by invariantal relations. We denote two arbitrary functions by the Roman capitals X, Y, and we consider the transformations effected by multiplying a given function by X and into Y.

This transformation changes the series of functions

$$f_1, f_2, f_3, \ldots f_1f_2^{-1}f_3, \ldots f_1f_2^{-1}f_3f_4^{-1}f_5 \ldots \quad . \quad . \quad . \quad . \quad . \quad (243)$$

into the series

$$Xf_1Y, Xf_2Y, Xf_3Y, \ldots Xf_1f_2^{-1}f_3Y, \ldots Xf_1f_2^{-1}f_3f_4^{-1}f_5Y \quad . \quad (244);$$

and we shall speak of this as the (XY) class.

The series

$$f_1^{-1}, f_2^{-1}, f_3^{-1}, \ldots f_1^{-1}f_2f_3^{-1}, \ldots f_1^{-1}f_2f_3^{-1}f_4f_5^{-1} \quad . \quad . \quad . \quad (245)$$

becomes

$$Y^{-1}f_1^{-1}X^{-1}, Y^{-1}f_2^{-1}X^{-1}, Y^{-1}f_3^{-1}X^{-1}, \ldots Y^{-1}f_1^{-1}f_2f_3^{-1}X^{-1}, \ldots$$
$$Y^{-1}f_1^{-1}f_2f_3^{-1}f_4f_5^{-1}X^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (246),$$

and this is the (Y^{-1}, X^{-1}) class.

The series

$$f_1f_2^{-1}, f_2f_3^{-1}, \ldots . f_1f_2^{-1}f_3f_4^{-1} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (247)$$

is the (XX^{-1}) class, transforming into the series

$$Xf_1f_2^{-1}X^{-1}, Xf_2f_3^{-1}X^{-1}, \ldots Xf_1f_2^{-1}f_3f_4^{-1}X^{-1} . \quad . \quad . \quad . \quad (248);$$

and finally the series

$$f_1^{-1}f_2, \quad f_2^{-1}f_3, \ldots \quad f_1^{-1}f_2f_3^{-1}f_4, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (249)$$

forms the ($Y^{-1}Y$) class, as it transforms into

$$Y^{-1}f_1^{-1}f_2Y, \quad Y^{-1}f_2^{-1}f_3Y, \ldots Y^{-1}f_1^{-1}f_2f_3^{-1}f_4Y \quad . \quad . \quad . \quad (250).$$

Inverse functions of the (XY) class belong to the ($Y^{-1}X^{-1}$) class, and conversely; inverse functions of the classes (XX^{-1}) or ($Y^{-1}Y$) belong to their own class, and so also do products and quotients of functions of these classes. The product of an (XY) function into a ($Y^{-1}X^{-1}$) function is an (XX^{-1}) function, and so on.

In like manner there are four classes for the conjugate functions, as appears on taking the conjugates of a typical function. The annexed scheme exhibits the eight classes, the conjugates being printed under their correspondents:—

$$\begin{array}{llll} (XY), & (Y^{-1}X^{-1}) & (XX^{-1}), & (Y^{-1}Y) \\ (Y'X') & (X'^{-1}Y'^{-1}) & (X'^{-1}X') & (Y'Y'^{-1}) \end{array} \quad . \quad . \quad . \quad . \quad . \quad . \quad (251).$$

60. When we deal with quadrics or complexes, or when the condition is imposed that self-conjugate functions remain self-conjugate, the classes of the conjugate type

coincide with those originally found, but in a different order. In this case Y is the conjugate of X, and the scheme (251) becomes

$$\begin{array}{cccc} (XX') & (X'^{-1}X^{-1}) & (XX^{-1}) & (X'^{-1}X') \\ (XX') & (X'^{-1}X^{-1}) & (X'^{-1}X') & (X\ X^{-1}) \end{array} \quad . \quad . \quad . \quad . \quad (252).$$

In this case the conjugate of a transformed function is the transformed function of the conjugate.

Again, in the general case, when $Y = X^{-1}$, the types of the upper row (251) merge in the single type (XX^{-1}), and the conjugates in the type $(X'^{-1}X')$.

Finally, all types unite in the single class (XX') when X is the inverse of its conjugate (Art. 47).

61. Covariant functions may be derived by the following general process, as well as by multiplication and division. For arbitrary scalars, t_1, t_2, t_3, &c.,

$$n_t(\Sigma tf)^{-1}[abc] = [\Sigma tf'a,\ \Sigma tf'b,\ \Sigma tf'c] = \Sigma t_1t_2t_3\ F_{123}[abc] \quad . \quad . \quad . \quad (253),$$

where n_t is the fourth invariant of Σtf, and where

$$F_{123}[abc] = \Sigma[f'_1a, f'_2b, f'_3c] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (254),$$

the summation in this last equation referring to permutation of the suffixes.

These functions belong to the $(Y^{-1}X^{-1})$ class, because

$$\Sigma[Y'f'_1X'a,\ Y'f'_2X'b,\ Y'f'_3X'c] = n_YY^{-1}F_{123}n_XX^{-1}[abc] \quad . \quad . \quad . \quad (255),$$

n_X and n_Y being the fourth invariants of X and Y.

In like manner functions of the (XY) class are obtainable in the form

$$f_{123}[abc] = \Sigma[F_1'a, F_2'b, F_3'c]\,;\ F_1'[abc] = [f_1a, f_1b, f_1c] \quad . \quad . \quad . \quad (256).$$

62. Although rather foreign to the subject of this paper, it may be as well to indicate the nature of the Hamiltonian quaternion invariants of a system of functions. It was stated in a paper on Quaternion Arrays* that these invariants are included in the quotient

$$\left\{\begin{array}{cccc} f_1a & f_1b & f_1c & f_1d \\ f_2a & f_2b & f_2c & f_2d \\ . & . & . & . \\ . & . & . & . \\ f_na & f_nb & f_nc & f_nd \\ a & b & c & d \end{array}\right\} \div (abcd) \quad . \quad . \quad . \quad . \quad . \quad (257),$$

formed by dividing a four-column array by $(abcd)$, each row of the array consisting of the results of operation by a single function on four arbitrary quaternions. Briefly,

* 'Trans. Roy. Irish Acad.,' vol. 32, p. 30.

a quaternion array may be defined as a function which vanishes if, and only if, the constituents of every row can be linearly connected by the same set of scalar multipliers. It is multiplied by a scalar if every constituent in a column is multiplied by that scalar; and the sign of the array is changed if contiguous columns are transposed.

These laws are precisely the laws which govern the function $(abcd)$, which is in fact a one-row array, so that if in (257) we replace any one of the four quaternions by any quaternion $xa + yb + zc + wd$, the quotient remains unchanged. The quotient is therefore an invariant in the Hamiltonian sense; it remains unchanged when the four quaternions a, b, c, d are operated on by the function Y.

If we regard the lowest row as consisting of the results of operating by the special linear function *unity* on a, b, c and d, and if we replace $f_1, f_2 \ldots f_n$ by $\mathrm{X}f_1\mathrm{Y}$, $\mathrm{X}f_2\mathrm{Y}, \ldots \mathrm{X}f_n\mathrm{Y}$ and unity in the last row by XY; to a factor, $n_{\mathrm{Y}}n_{\mathrm{X}}$, the quotient becomes the corresponding quotient for the system of functions

$$\mathrm{X}f_1\mathrm{X}_1^{-1}, \mathrm{X}f_2\mathrm{X}_2^{-1}, \ldots \mathrm{X}f_n\mathrm{X}^{-1}.$$

SECTION XI.

The Numerical Characteristics of Certain Curves and Assemblages of Points.

63. In order to facilitate future investigations, we shall determine the numerical characteristics of certain curves and systems of points which frequently occur.

Using the symbol Q_n to denote a homogeneous quaternion function of q of the order M_n, it appears from SALMON's chapter on the "Order of Restricted Systems of Equations" in his 'Modern Higher Algebra,' that

$$\{Q_1Q_2\} = 0, \quad \text{or} \quad t_1Q_1 + t_2Q_2 = 0 \quad \ldots \ldots \quad (258)$$

represents a system of points whose number is

$$M_1^3 + M_1^2M_2 + M_1M_2^2 + M_2^3 \quad \ldots \ldots \quad (259).$$

64. In like manner the chapter cited enables us to write down the order of the curve represented by

$$[Q_1Q_2Q_3] = 0, \quad \text{or} \quad t_1Q_1 + t_2Q_2 + t_3Q_3 = 0 \quad \ldots \ldots \quad (260);$$

but as it is desirable to determine also its rank and the number of its apparent double points, we shall adopt a different method.

The quaternions a and b being arbitrary, the identity

$$Q_1(Q_2Q_3ab) + Q_2(Q_3abQ_1) + Q_3(abQ_1Q_2) + a(bQ_1Q_2Q_3) + b(Q_1Q_2Q_3a) = 0 \quad . \quad (261),$$

shows that the two surfaces

$$(aQ_1Q_2Q_3) = 0, \quad (bQ_1Q_2Q_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (262)$$

intersect in the curve (260), and also in a complementary curve common to the three surfaces

$$(abQ_2Q_3) = 0, \quad (abQ_3Q_1) = 0, \quad (abQ_1Q_2) = 0 \quad . \quad . \quad . \quad . \quad (263);$$

for when (262) is satisfied, the identity shows that either (260) or (263) must be satisfied.

Let m denote the order of the curve (260); then the order of the complementary is

$$(M_1 + M_2 + M_3)^2 - m = m' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (264),$$

the orders of the two surfaces (262) being $M_1 + M_2 + M_3$.

Again, considering the intersection of the second and third surfaces (263), it follows from the identity that they intersect in the complementary curve and in the new curve

$$[Q_1ab] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (265);$$

and because the orders of the surfaces are $M_1 + M_3$ and $M_1 + M_2$, the order $m_{\prime}$ of this new curve is connected with m' by the relation

$$(M_1 + M_2)(M_1 + M_3) - m' = m_{\prime} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (266).$$

Again, writing down the identity

$$a(bcqQ_1) + b(cqQ_1a) + c(qQ_1ab) + q(Q_1abc) + Q_1(abcq) = 0 \quad . \quad (267),$$

in which q is the variable quaternion, while a, b and c are constants, it appears exactly as before that the surfaces

$$(abqQ_1) = 0, \quad (abcQ_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (268),$$

of orders $M_1 + 1$ and M_1, intersect in the curve (265) and in a complementary curve which is obviously the complete intersection of the surfaces

$$(abcq) = 0, \quad (abcQ_1) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (269);$$

that is, a plane and a surface of order M_1.

Now the relations*

* SALMON's 'Geometry of Three Dimensions,' Arts. 345, 346.

$$2(h-h') = (m-m')(\mu-1)(\nu-1), \quad r-r' = (m-m')(\mu+\nu-2) \quad . \quad (270)$$

connect the number of apparent double points (h) and the rank (r) of a curve with those of its complementary in the intersection of two surfaces of orders μ and ν. But we know the characteristics of the plane curve (269) to be

$$m_{,}' = M_1, \quad r_{,}' = M_1(M_1-1), \quad h_{,}' = 0 \quad . \quad . \quad . \quad . \quad . \quad (271);$$

and hence we find the characteristics of its complementary (265),

$$m_{,} = M_1^2, \quad r_{,} = 2M_1^2(M_1-1), \quad 2h_{,} = M_1^2(M_1-1)^2 \quad . \quad . \quad (272);$$

and these in turn give the characteristics for the curve m',

$$m' = \Sigma_2; \quad r' = \Sigma_2(\Sigma_1 - 2) + \Sigma_3; \quad 2h' = \Sigma_2(\Sigma_2 - \Sigma_1 + 1) - \Sigma_3 \quad . \quad (273),$$

and, finally, for the original curve (260) we have

$$m = \Sigma_1^2 - \Sigma_2; \quad r = 2\Sigma_1^3 - 3\Sigma_1\Sigma_2 + \Sigma_3 - 2(\Sigma_1^2 - \Sigma_2);$$
$$2h = (\Sigma_1^2 - \Sigma_2)^2 - (2\Sigma_1^3 - 3\Sigma_1\Sigma_2 + \Sigma_3) + (\Sigma_1^2 - \Sigma_2) \quad . \quad (274),$$

where Σ_1, Σ_2 and Σ_3 are the sum, the sum of the products in pairs, and the product of the three quantities M_1, M_2 and M_3.

As examples, for the twisted cubic

$$[f_1 q f_2 q a] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (275),$$

$M_1 = M_2 = 1$, $M_3 = 0$, and $\Sigma_1 = 2$, $\Sigma_2 = 1$, $\Sigma_3 = 0$, so that $m = 3$, $r = 4$, $h = 1$. For the curve

$$[f_1 q f_2 q f_3 q] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (276),$$

$\Sigma_1 = 3$, $\Sigma_2 = 3$, $\Sigma_3 = 1$; and $m = 6$, $r = 16$, $h = 7$.

These numbers admit of course of simple verification.*

65. In like manner proceeding one step further we calculate the characteristics of the curve common to the five surfaces obtained by equating to zero the coefficients in the identity

$$Q_1(Q_2Q_3Q_4Q\ ) + Q_2(Q_3Q_4Q_5Q_1) + Q_3(Q_4Q_5Q_1Q_2) + Q_4(Q_5Q_2Q_2Q_3) + Q_5(Q_1Q_2Q_3Q_4)$$
$$= 0 \quad . \quad . \quad (277)$$

to be

$$m = \Sigma M_1M_2, \quad r = \Sigma M_1 \Sigma M_1M_2 + \Sigma M_1M_2M_3 - 2\Sigma M_1M_2 \quad . \quad . \quad . \quad (278);$$

this curve being the complementary of (260) for the fourth and fifth surfaces.

The curve common to the five surfaces may be conveniently designated by the equation in double brackets

* The expression for the rank of a curve, 'Modern Higher Algebra,' Art. 284, seems to require modification.

$$((Q_1Q_2Q_3Q_4Q_5)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (279)$$

which is intended to denote that every set of four of the included quaternions is linearly connected.

66. For the number of points common to the surfaces whose equations are obtained by deleting two of the quaternions included in triple brackets

$$(((Q_1Q_2Q_3Q_4Q_5Q_6))) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (280),$$

SALMON's formula ('Modern Higher Algebra') gives

$$N = \Sigma M_1M_2M_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (281).$$

67. To complete the scheme, we may regard the equation

$$[[Q_1Q_2Q_3Q_4]] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (282),$$

as requiring the four quaternions Q_1, Q_2, Q_3, Q_4 to be collinear; or the four curves (260), obtained by omitting one quaternion, to have common points. If these points exist they satisfy the equation (compare (279))

$$((aQ_1Q_2Q_3Q_4)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (283),$$

or lie on the complementary common to the five surfaces.

A curve meets its complementary ('Geometry of Three Dimensions,' Art. 346) in

$$t = m(\mu + \nu - 2) - r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (284)$$

points, and in particular for the curve $[Q_1Q_2Q_3]$ and the two surfaces $(aQ_1Q_2Q_3) = 0$, $(Q_1Q_2Q_3Q_4) = 0$, we find the number to be (compare (274))

$$t_4 = \Sigma_1\Sigma_2 - \Sigma_3 + M_4(\Sigma_1^2 - \Sigma_2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (285).$$

These points are generally variable with the arbitrary quaternion a.

Again, the surface

$$(aQ_1Q_2Q_3)(bQ_1Q_2Q_4) + u(aQ_1Q_2Q_4)(bQ_1Q_2Q_3) = 0 \quad . \quad . \quad . \quad . \quad (286)$$

intersects $(Q_1Q_2Q_3Q_4) = 0$ in $[Q_1Q_2Q_3] = 0$, $[Q_1Q_2Q_4] = 0$, and in the complementary corresponding to b. When we seek the intersection of the curve $[Q_1Q_2Q_3] = 0$ with its complex complementary on this surface, the number of points is found to be $2t_4 + M_1^3 + M_1^2M_2 + M_1M_2^2 + M_2^3$, and these can all be accounted for by (285) and (259).

We can also in this manner determine the points common to the two complementaries (283) answering to a and b to be $\Sigma M_1M_2M_3$, employing the characteristics (278), and putting $M_5 = 0$.

SECTION XII.

On the Geometrical Relations depending on Two Functions and on the Four Functions f, f', f_0 and $f_{\prime}$.

68. We devote this section to the study of the geometrical relations connecting a function f with its conjugate f', its self-conjugate part f_0 and its non-conjugate part $f_{\prime}$ (Art. 9), and to the relations connecting a pair of arbitrary functions f_1 and f_2.

The quadric

$$Sqfq = Sqf'q = Sqf_0q \quad \ldots \ldots \ldots \quad (287)$$

is the locus of a point which is conjugate with respect to the unit sphere to its correspondent in each of the transformations due to f, f' and f_0.

The linear complex

$$Spf_{\prime}q = 0, \text{ or } Spfq = Sqfp, \text{ or } Spf'q = Sqf'p \quad \ldots \quad (288),$$

may be written in the form (compare p. 223).

$$S\text{PQ}'Sf\text{Q} = S\text{QP}'Sf\text{P}, \quad (\text{P}Sp = p, \ \text{P}'Sf\text{P} = f\text{P}) \quad \ldots \ldots \quad (289),$$

which expresses that the product of the perpendiculars from Q′, the derived of one point Q, and from the centre of reciprocation on the polar plane of another point P with respect to the unit sphere, multiplied by the perpendicular ($Sf\text{Q}$) from Q on the plane which is projected to infinity by the transformation, is equal to the corresponding product of three perpendiculars found by interchanging P and Q. This property is also true when f is replaced by its conjugate f'.

The equation of the complex may also be regarded as representing the assemblage of lines converted by $f_{\prime}$ into conjugate lines with respect to the unit sphere.

69. In order to determine the four lines common to the quadric and the linear complex, observe that the point of contact $(f_0^{-1}h)$ of a plane $Shq = 0$ with the quadric must also be the point of concourse $(f_{\prime}^{-1}h)$ of the lines of the complex in that plane, in order that the plane may contain lines common to the two assemblages. Therefore the points e in which the pairs of common lines intersect satisfy the equations

$$e = f_{\prime}^{-1}h = u^{-1}f_0^{-1}h, \text{ or } h = f_{\prime}e = uf_0 e \quad . \quad . \quad . \quad . \quad . \quad (290).$$

Thus four points e are determined, the united points of the function $f_0^{-1}f_{\prime}$.

It appears, as in Art. 12, that the latent roots of this function are equal and opposite, and that the united points form a quadrilateral on the quadric.

Otherwise, the invariants of $f_0^{-1}f_{\prime}$ and of $f_{\prime}f_0^{-1}$ are identical (Art. 23), and these functions satisfy the same symbolic quartic; and because their conjugates, $-f_{\prime}f_0^{-1}$ and $-f_0^{-1}f_{\prime}$ likewise satisfy the same quartic, it must be of the form

$$(f_0^{-1}f_{\prime})^4 + N''(f_0^{-1}f_{\prime})^2 + N = 0, \text{ or } ((f_0^{-1}f_{\prime})^2 - u_1^2)\,((f_0^{-1}f_{\prime})^2 - u_2^2) = 0 \quad . \quad (291).$$

Hence the lines in question are determined on solution of a quadratic equation.

When these four points e_1, e'_1, e_2, e'_2 are taken as points of reference,* so that

$$q = \frac{xe_1 + ye'_1}{\sqrt{Se_1f_0e'_1}} + \frac{ze_2 + we'_2}{\sqrt{Se_2f_0e'_2}},\ p = \frac{x'e_1 + y'e'_1}{\sqrt{Se_1f_0e'_1}} + \frac{z'e_2 + w'e'_2}{\sqrt{Se_2f_0e'_2}} \quad . \quad . \quad (292)$$

the equations of the quadric and complex may by the aid of (290) (compare again Art. 12) be reduced to the forms

$$xy + zw = 0 \qquad u_1(xy' - x'y) + u_2(zw' - z'w) = 0 \,. \quad . \quad . \quad . \quad (293).$$

70. The locus of points whose correspondents are in perspective with a fixed point a is the twisted cubic

$$fq + tq = a \quad \text{or} \quad [fq, q, a] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (294),$$

and the locus of lines which pass through a fixed point a and connect a point and its correspondent is the cone

$$fq + tq = xfa + ya \quad \text{or} \quad (fqqfaa) = 0. \quad . \quad . \quad . \quad . \quad . \quad (295).$$

* Observe that these four points e are the only points for which

$$fq \doteq f'q \doteq f_0q \doteq f_{\prime}q,$$

the signs $\doteq$ being used to denote equality when the quaternions are multiplied by a suitable factor. For vector functions

$$\phi\rho = \phi'\rho = \phi_0\rho$$

only when $\rho = \epsilon$, where ϵ is the spin-vector.

The complex of lines connecting points and their correspondents has for its equation*

$$(fppfqq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (296);$$

and the locus of points whose connectors to correspondents intersect a fixed line ab is the quadric surface

$$(fqqab) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (297).$$

The reciprocal of the complex (296) is the complex of the conjugate

$$(f'ppf'qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (298);$$

for the line $p'q'$ is reciprocal to the line p, fp if $Spp' = Spq' = Spf'p' = Spf'q' = 0$, which requires p', q', $f'p'$, $f'q'$ to be coplanar.

The formulæ of this article comprise many theorems with respect to the normals of confocal quadrics. It may also be observed that the complex (296) is unchanged when f is replaced by $(f+x)(f+y)^{-1}$.

71. *An arbitrary quadric has eight generators which connect a point and its correspondent in an arbitrary transformation.* This is the extension of HAMILTON'S celebrated theory of the umbilical generators. (Compare Art. 40.)

The conditions that the line $q = fa + sa$ should be a generator of the arbitrary quadric surface

$$SqFq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (299)$$

are

$$SaFa = 0, \quad Sa(f'F + Ff')a = 0, \quad Saf'Ffa = 0 \quad . \quad . \quad . \quad . \quad (300);$$

so that we can determine eight points a as the intersections of three known quadrics, and the lines joining these points to their correspondents are the common generators of the complex and the quadric.

Four of these lines are generators of one system of the quadric and four of the other system.

Four of the lines must belong to one system of generators. Let these be determined by the points a_1, a_2, a_3, a_4. The condition that the line pq should meet the line $a_1 fa_1$ is

$$(pqa_1fa_1) = 0 \quad \text{or} \quad S(pq)[a_1fa_1] + S[pq](a_1fa_1) = 0 \quad . \quad . \quad (301);$$

and because any line which meets three of these four lines likewise meets the fourth, we must have for proper selection of the weights

$$(a_1fa_1)+(a_2fa_2)+(a_3fa_3)+(a_4fa_4) = 0, \; [a_1fa_1]+[a_2fa_2]+[a_3fa_3]+[a_4fa_4] = 0 \quad (302).$$

* When we refer p and q to the united points of f, the equation of this complex takes the forms

$$\Sigma(t_2t_3 + t_1t_4)(yz' - y'z)(xw' - x'w) = 0, \; \Sigma(t_2 - t_3)(t_1 - t_4)(yzx'w' + y'z'xw) = 0,$$

where

$$p = xa + yb + zc + wd, \quad q = x'a + y'b + z'c + w'd.$$

A vector equation may also be employed, for if we put $p = 1 + \alpha$, $q = p + \rho$, the equation of the complex may be replaced by

$$(f+t)\,\rho = u\,(f+s)(1+\alpha), \quad \text{or} \quad \rho = v\,(V\,(f+t)^{-1}(1+\alpha) - \alpha S\,(f+t)^{-1}(1+\alpha)),$$

when we eliminate s by separating the scalar and vector parts after inversion of $f+t$.

Hence the eight points common to three of the quadrics

$$(qfqa_nfa_n) = 0 \qquad (n = 1, 2, 3, \text{ or } 4) \quad . \quad . \quad . \quad . \quad . \quad (303)$$

are likewise common to the fourth. But four of these points are the united points of the function f, while the remaining four determine (297) four lines of the complex (296) which meet the four generators. These four lines are common to the quadric and the complex, and make up with the other four the complete system of eight lines.

In accordance with (302) we may write for the two sets of four lines*

$$\{a_1fa_1\} + \{a_2fa_2\} + \{a_3fa_3\} + \{a_4fa_4\} = 0,$$
$$\{a'_1fa'_1\} + \{a'_2fa'_2\} + \{a'_3fa'_3\} + \{a'_4fa'_4\} = 0 \quad . \quad . \quad . \quad . \quad (304),$$

and it may be remarked that a direct interpretation of (302) is that four equilibrating forces can be placed along the lines of either set, for the first equation (302) expressed that the resultant of four forces vanishes, and the second requires their moment with respect to the centre of reciprocation to be zero† (see (33), p. 230).

72. The locus of the united points of all functions of the system

$$(x'f + y'f' + z')^{-1}(xf + yf' + z) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (305)$$

is the curve

$$[fqf'qq] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (306);$$

and this curve (276) is a sextic whose rank is 16, and the number of whose apparent double points is 7.

If q is a united point of a function (305) and t the corresponding latent root, we obviously have

$$(x - tx')fq + (y - ty')f'q + (z - tz')q = 0 \quad . \quad . \quad . \quad . \quad (307),$$

whence (306) follows immediately.

The sextic curve is evidently the locus of united points of the conjugates $(xf' + yf + z)$ of functions $xf + yf' + z$, but it is not the locus of united points of conjugates of functions of the general type (305).

In the following articles we shall consider some part of the theory of two arbitrary functions f_1 and f_2, as it is partially applicable to the subject under discussion.

73. The loci of the united points of all functions of the two systems

$$(x'f_1 + y'f_2 + z')^{-1}(xf_1 + yf_2 + z) \text{ and } (x'f_1' + y'f_2' + z')^{-1}(xf_1' + yf_2' + z) \quad (308)$$

are respectively the sextic curves

$$[f_1qf_2qq] = 0, \qquad [f_1'qf_2'qq] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (309).$$

These two curves unite in the special case of $f_2 = f_1'$. The first is the locus of the united points of the system $xf_1 + yf_2 + z$, and the second is the corresponding locus for the conjugate system.

* In the notation of arrays $\Sigma\{p_nq_n\} = 0$ implies $\Sigma(p_nq_n) = \Sigma[p_nq_n] = 0$.

† If $a_n = \mathrm{A}_n\mathrm{S}a_n$, $fa_n = \mathrm{B}_n\mathrm{S}fa_n$, $\mathrm{A}_n = 1 + \alpha_n$, $\mathrm{B}_n = 1 + \beta_n$, the equations (302) become

$$\Sigma(\beta_n - \alpha_n)\,\mathrm{S}a_n\mathrm{S}fa_n = 0; \; \Sigma\mathrm{V}\alpha_n\beta_n\mathrm{S}a_n\mathrm{S}fa_n = 0.$$

The locus of the united planes of the system $xf_1 + yf_2 + z$ *is the reciprocal of the conjugate sextic.*

By the conjugate sextic we mean the second curve (309), and the proposition is obvious when we reflect that a united plane of a function is the reciprocal of the corresponding united point of its conjugate (Art. 8).

The united plane of a function of the system $xf_1 + yf_2 + z$ *cuts the sextic in three united points and in three other collinear points.*

The equation of a united plane of the function $xf_1 + yf_2 + z$ is $Sa'q = 0$, where a' is a united point of the conjugate. Writing the equation of the sextic in the form

$$x'f_1q + y'f_2q + z'q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (310),$$

and expressing that q lies in the plane, the result is

$$Sq\,(x'f_1'a' + y'f_2'a') = 0, \text{ or } Sq\,((x' - sx)\,f_1'a'_1 + (y' - sy)f_2'a') = 0 \quad (311),$$

where s is arbitrary, because $xf_1'a' + yf_2'a' + za' = ta'$.

Hence either $x' = x$, $y' = y$, and q is a united point of the function, or else

$$Sqa' = Sqf_1'\,a' = Sqf_2'a' = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (312);$$

and the three remaining points are collinear.

In particular for the functions f, f', f_0, f_i, the polar plane with respect to the quadric and the plane of rays of the complex, corresponding to the reciprocal of a united plane of the function f, intersect in that united plane; and their common line is a three-point chord of the sextic (306).

74. Knowing the rank and number of apparent double points of the sextic, its characteristics are

$$r = 16,\ m = 6,\ n = 30,\ \alpha = 48,\ \beta = 0,\ x = 96,\ y = 72,\ g = 355,\ h = 7 \quad (313),$$

as may be verified by the formulæ printed in Arts. 326–7 of SALMON's 'Geometry of Three Dimensions.' Also the deficiency of the curve is $D = 3$.

These numbers apply reciprocally to the developable of the last article generated by the united planes. Thus the order of its cuspidal curve is 30, and six united planes pass through an arbitrary point, while sixteen pass through a line.

Through a united point the six united planes consist of the three planes which are united planes of the function possessing the united point, and three other planes intersecting in a common line (compare (312)) which is the reciprocal of a three-point chord of the second sextic.

75. *The triple chords of the sextic generate a surface of the eighth order.*

The three-point chords of a curve generate a surface of order ('Three Dimensions,' Art. 471)

$$\tfrac{1}{6}\,(m - 2)\,(6h + m - m^2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (314),$$

and this reduces to 8 in the present case.

The characteristics of the cone, whose vertex is a point on the sextic and which contains the sextic, are deducible from the data of Art. 330 of the 'Geometry of Three

Dimensions.' The cone has $h - m + 2$ double edges, and consequently three triple chords pass through an arbitrary point on the sextic. The sextic is thus a triple curve on the regulus of triple chords, and the surface has no other multiple line.

76. There is yet another quadratic complex of importance in the study of a pair of functions. A point a is transformed into $xf_1a + yf_2a$ by the operation of $xf_1 + yf_2$, and as x and y vary, the locus of the transformed point is a line which we shall call the *satellite* of a.

The satellites generate the complex

$$(f_1^{-1}p\,f_2^{-1}p\,f_1^{-1}q\,f_2^{-1}q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (315),$$

and the form of this equation should be compared with (296) and (298). There is also the complex of conjugate satellites obtained by replacing f_1 and f_2 by their conjugates, but when the functions are self-conjugate, or when one is the conjugate of the other, the two complexes combine into one. For the functions f and f' this is

$$(f^{-1}p\,f'^{-1}p\,f^{-1}q\,f'^{-1}q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (316).$$

The four points $fa, f'a, f_0a, f_ia$ form a harmonic range on the satellite of the point a. There are also harmonic properties connecting pencils of planes $\mathrm{S}qfa = 0$, $\mathrm{S}qf'a = 0$, $\mathrm{S}qf_0a = 0$, $\mathrm{S}qf_ia = 0$; and it may be verified that these four planes intersect in a satellite for the inverse functions. This we shall prove for the general case.

The reciprocal of the complex of satellites is the complex of the conjugate satellites for the inverse functions.

If p and q are any two points on the reciprocal of the satellite of a,

$$\mathrm{S}pf_1a = \mathrm{S}pf_2a = 0, \quad \mathrm{S}qf_1a = \mathrm{S}qf_2a = 0 \quad . \quad . \quad . \quad . \quad . \quad (317),$$

and on taking conjugates we see that the four points $f_1'p$, $f_2'p$, $f_1'q$, $f_2'q$ are co-planar, so that

$$(f_1'p\,f_2'p\,f_1'q\,f_2'q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (318).$$

The locus of points whose satellites meet the line ab is the quadric surface (compare (297))

$$(f_1q\,f_2q\,ab) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (319).$$

77. The satellite of a point which describes a line $q = a + tb$ constructs one system of generators of the quadric

$$q = (f_1 + sf_2)(a + tb) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (320),$$

but the regulus degrades into a system of lines enveloping a conic whenever

$$(f_1af_2af_1bf_2b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (321),$$

that is, whenever the line belongs to the reciprocal of the complex of conjugate satellites (318).

The conic is co-planar with the line when the further conditions

$$(abf_1af_1b) = 0, \quad (abf_2af_2b) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (322),$$

are satisfied (compare (296), (298)).

But when we are given, as here, a series of tangents to a conic homographic with a series of points on a line in its plane, in three cases a tangent passes through its corresponding point; and evidently when a point lies on its satellite, it also lies on the sextic $[f_1 q f_2 q q] = 0$; so the line under discussion is a triple chord of the sextic.

It seems worth while noticing (compare Art. 66) the remarkable equation

$$(((a, b, f_1 a, f_1 b, f_2 a, f_2 b))) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (323)$$

of the *assemblage of triple chords of the sextic*, for this equation is equivalent to (321) and (322).

78. Again, in an arbitrary plane $Slq = 0$, it is generally possible to find one point p whose satellite lies in the plane. The conditions are

$$Slp = 0, \quad Slf_1 p = 0, \quad Slf_2 p = 0, \quad \text{so} \quad p = [l, f_1' l, \; f_2' l] \quad . \quad . \quad . \quad (324);$$

and the point is determinate unless the reciprocal of the plane lies on the conjugate sextic (Art. 73), or, in other words, unless the plane is a united plane for some function of the system. In this case (compare (312)) there exists a line locus for points p whose satellites lie in the plane.

This is precisely the case of the last article, so when the envelope of satellites is a conic co-planar with the line, the plane is a united plane.

79. For an arbitrary plane, the locus of points whose deriveds by $f_1 + xf_2$ remain in the plane is the line of intersection of $Sl(f_1 + xf_2) q = 0$ or $Sq(f_1' + xf_2') l = 0$ with the given plane $Slq = 0$. All these lines pass through the point p, which may be called the *focus* of the plane.

Assuming an arbitrary point p to be a focus, the plane of which it is the focus is (compare (324)) the reciprocal of the point

$$l = [pf_1 pf_2 p] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (325).$$

The relation between a focus and the reciprocal of the plane is of the same nature as the correspondence discussed in Section XIX. (compare (526) with (324)).

The points whose satellites pass through a given point a lie on a twisted cubic

$$[af_1 q f_2 q] = 0,$$

and the locus of points whose satellites lie in a plane is a right line. The satellite of a point q and the plane $Slq = 0$ pierces the plane in the point

$$q_{\prime} = f_1 q Slf_2 q - f_2 q Slf_1 q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (326),$$

and from this quadratic transformation connecting the points q and $q_{\prime}$, it follows that q (or $q_{\prime}$) describes a conic when $q_{\prime}$ (or q) describes a right line. In the former case the conics pass through the focus of the plane. Thus again an arbitrary line qq' meets the satellites of two points on the line (compare (320)).

It would take too long to explain the various geometrical relations in the plane

$Slq = 0$, but subjects such as that just mentioned may be readily investigated by writing

$$q = xa + yb + zc, \quad q' = x'a + y'b + z'c,$$

where a, b and c are any three points in the plane. Then the array

$$\{qq'\} = \lambda \{bc\} + \mu \{ca\} + \nu \{ab\} \quad \text{if} \quad \lambda = yz' - y'z, \ \mu = zx' - z'x, \ \nu = xy' - x'y \quad . \quad (327)$$

and

$$\{f_t q, f_t q'\} = \lambda \{f_t b, f_t c\} + \mu \{f_t c, f_t a\} + \nu \{f_t a, f_t b\} \quad . \quad . \quad . \quad (328),$$

if

$$f_t = f_1 + tf_2.$$

Hence (compare (301) and (296)) the line qq' joins a point to its correspondent in the transformation produced by f_t if

$$\Sigma\lambda^2 (bcf_t bf_t c) + \Sigma\mu\nu \{(caf_t af_t b) + (abf_t cf_t a)\} = 0 \quad . \quad . \quad . \quad . \quad (329).$$

This equation may be regarded as the tangential equation of a conic involving a parameter t quadratically. For six values of t the equation represents a pair of points—one point of each pair being one of the six points in which the plane meets the critical sextic, and the second point being the intersection of the plane with the *line* into which the plane is transformed by the function $(f_t - s)$ which destroys the aforesaid point (compare Art. 14, I).

In a united plane, the theory is simpler. Let a, b, c be the united points in the plane, united points of f_1. Then (327) and (328) become

$$\{qq'\} = \lambda \{bc\} + \mu \{ca\} + \nu \{ab\},$$

$$\{(f_1 + tf_2)\, q, \ (f_1 + tf_2)\, q'\} = \lambda \{t_2 b + tf_2 b, \ t_3 c + tf_2 c\} + \mu \{t_3 c + tf_2 c, \ t_1 a + tf_2 a\}$$
$$+ \nu \{t_1 a + tf_2 a, \ t_2 b + tf_2 b\} \quad . \quad (330);$$

and we get the conics

$$t^2 \{\Sigma\lambda^2 (bcf_2 bf_2 c) + \Sigma\mu\nu [(caf_2 af_2 b) + (abf_2 cf_2 a)]\} - t\Sigma (t_2 - t_3)\, \mu\nu\, (abcf_2 a) = 0 \quad . \quad (331).$$

In this case the system of conics is inscribed to a common quadrilateral.

The conic enveloped by the satellites is

$$\Sigma\lambda^2 t_1 (bcf_2^{-1} bf_2^{-1} c) + \Sigma\mu\nu [t_2 (caf_2^{-1} af_2^{-1} b) + t_3 (abf_2^{-1} cf_2^{-1} a)] = 0,$$

or

$$\Sigma\lambda^2 t_1 (bcf_2 bf_2 c) + \Sigma\mu\nu [t_2 (abf_2 cf_2 a) + t_3 (caf_2 af_2 b)] = 0 \quad . \quad . \quad . \quad . \quad . \quad (332).$$

80. More particularly for the functions $ff'f_0 f_p$ in a united plane of f, the united points a, b, c form a triangle (I) in perspective with the triangle (II) of the traces of the united planes of the conjugate; for these planes are

$$Sqa = 0, \quad Sqb = 0, \quad Sqc = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (333);$$

and the centre of perspective is given by

$$SqaSbc = SqbSca = SqcSab \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (334);$$

while corresponding sides intersect in the points

$$b\mathrm{S}ca - c\mathrm{S}ab, \quad c\mathrm{S}ab - a\mathrm{S}bc, \quad a\mathrm{S}bc - b\mathrm{S}ca \quad . \quad . \quad . \quad . \quad (335).$$

The point of concourse of lines of the linear complex in the plane is $f_{\prime}^{-1}[abc]$, or,

$$(t_2 - t_3)a\mathrm{S}bc + (t_3 - t_1)b\mathrm{S}ca + (t_1 - t_2)c\mathrm{S}ab \quad . \quad . \quad . \quad . \quad (336),$$

since this point is the intersection of the planes

$$\mathrm{S}qf_{\prime}a = 0, \quad \mathrm{S}qf_{\prime}b = 0, \quad \mathrm{S}qf_{\prime}c = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (337),$$

for which the united points are points of concourse. This point lies on the axis of perspective (335), and the equation of that axis may be written in the form

$$q = (f - t)f_{\prime}^{-1}[abc] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (338).$$

The three lines of the complex which pass through the united points intersect the sides of the triangle (I) in a triangle (III) in perspective with (I), and through the vertices of this third triangle pass the polars of the united points with respect to the quadric $\mathrm{S}qf_0q = 0$, and the traces of these planes form a triangle (IV) likewise in perspective with (I).

SECTION XIII.

The System of Quadrics $\mathrm{S}q\dfrac{f+s}{f+t}q = 0$, and some Questions relating to Poles and Polars.

81. In this section we shall notice some properties of the system of quadrics

$$\mathrm{S}q\frac{f+s}{f+t}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (339).$$

The self-conjugate function f in this homographic system may be supposed reduced to the type noticed in Art. 28, for by a linear transformation the symbolic quartic may be reduced in three ways to the form

$$f^4 + \mathrm{N}''f^2 + \mathrm{N} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (340).$$

The system (339) is its own reciprocal, and it includes confocal and concyclic

systems. If a is the pole of the plane $Sbq = 0$ with respect to one of the quadrics, a and b are connected by the equation

$$b = \frac{f+s}{f+t}a, \text{ or } (f+t)b = (f+s)a, \text{ or } a = \frac{f+t}{f+s}b \quad . \quad . \quad . \quad (341).$$

Given b, the locus of a is a twisted cubic if s alone varies, a right line if s is constant, and a quadric

$$(afa\ bfb) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (342)$$

when s and t are both variable. (Compare Art. 70.)

The points of contact of the plane with quadrics of the system are found by adding the condition $Sab = 0$, when we find three points, one point or a conic locus.

A generalized normal joins a point to the reciprocal of its tangent plane, thus for u variable,

$$q = \frac{f+u}{f+t}a, \qquad \text{when} \qquad Sa\frac{f+s}{f+t}a = 0 \quad . \quad . \quad . \quad . \quad (343)$$

is the generalized normal at the point a; or deleting the condition and allowing t and u to vary, we have the equation of the assemblage of normals through the point a, and when a itself varies, we see that (342) represents the complex of normals to the system.

82. In general, two quadrics $s_1\ t_1$ and $s_2\ t_2$ intersect in a curve through which no third quadric of the system can pass, but when $t_1 = t_2$, an infinite number of the quadrics intersect in the curve. This follows from the consideration that

$$Sq \, . \, \frac{x(f+s_1)(f+t_2) + y(f+s_2)(f+t_1)}{(f+t_1)(f+t_2)} q = 0 \quad . \quad . \quad . \quad . \quad (344)$$

is the general equation of a quadric through the curve; and a factor will not cancel unless $t_1 = t_2$.

If q is any point on the curve of intersection, the poles of the tangent planes at that point with respect to some third quadric of the system will be conjugate to that quadric if

$$Sq\frac{(f+s_1)(f+s_2)(f+t_3)}{(f+t_1)(f+t_2)(f+s_3)}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (345).$$

In order that this may be the case for every point on the curve, the factor $f+s_3$ must cancel. Thus we must have s_3 equal s_1, s_2 or t_3. But further, on comparison with (344), it appears that the third quadric must coincide with one of the others, or else that $t_3 = s_3$ and $s_1 = s_2$.

This theory embraces the laws of confocals, their orthogonal section, and the property that the pole of the tangent plane to one, at a point of intersection with a second, taken with respect to the second, lies in its tangent plane at the point.

83. More generally, given any three quadrics

$$Sqf_1q = 0,\ Sqf_2q = 0,\ Sqf_3q = 0 \quad \ldots \ldots \quad (346);$$

take the polar planes of a point q with respect to the first and second, and the poles of these planes with respect to the reciprocal of the third; these poles are conjugate to that reciprocal provided the point lies upon the quadric

$$Sqf_2f_3f_1q = 0 \quad \ldots \ldots \quad (347).$$

If the quadrics have a common self-conjugate tetrahedron with the quadric of reciprocation, the three functions have the same united points, and are consequently commutative; and the three surfaces (347) obtainable for different selections of the quadrics (346) are identical.

84. Before leaving this subject, it may be of interest to show how the invariant condition that three quadrics should be polar quadrics of a cubic presents itself.

We have, if the quadrics are polars of the cubic $F(qqq) = 0$,

$$Sqf_1q = F(aqq),\ Sqf_2q = F(bqq),\ Sqf_3q = F(cqq) \quad \ldots \quad (348),$$

if a, b, c are the poles. Hence

$$Sqf_1b = Sqf_2a;\ Sqf_2c = Sqf_3b;\ Sqf_3a = Sqf_1c \quad \ldots \quad (349);$$

and on identifying the planes

$$f_1b = f_2a;\ f_2c = f_3b;\ f_3a = f_1c \quad \ldots \ldots \quad (350);$$

so that

$$a = f_2^{-1}f_1f_3^{-1}f_2f_1^{-1}f_3a \quad \ldots \ldots \quad (351);$$

and the function $f_2^{-1}f_1f_3^{-1}f_2f_1^{-1}f_3$ must have one latent root equal to unity.

SECTION XIV.

Properties of the General Surface.

85. If Q is a homogeneous and scalar function of a variable quaternion q of order m, the equation

$$Q = 0 \quad \ldots \ldots \quad (352)$$

represents a surface. We shall write generally for any differential

$$dQ = mSp\,dq. \quad \ldots \ldots \ldots \quad (353),$$

where p is a homogeneous quaternion function of q and of the order $m-1$. Since p is a determinate function of q, q may be regarded as a function of p; and using EULER's theorem for homogeneous functions we have

$$Q = Spq = P \quad \ldots \ldots \ldots \quad (354)$$

where P is the function of p into which Q transforms.

86. Again, we shall write generally for the differential of the quaternion p regarded as a function of q,

$$dp = (m-1)f\,dq \quad \ldots \ldots \ldots \quad (355)$$

where $f\,dq$ is a linear function of dq, involving q homogeneously in the order $m-2$.

This function is self-conjugate, for taking two successive and independent differentials of Q,

$$\begin{aligned} d'\,dQ &= mSp\,d'\,dq + m(m-1)\,S\,.\,f d'q\,.\,dq \\ = dd'Q &= mSp\,dd'q + m(m-1)\,S\,.\,f\,dq\,.\,d'q\,. \quad \ldots \ldots \quad (356); \end{aligned}$$

and because the differentials are independent,

$$d'dq = dd'q, \text{ and therefore } Sdq\,f\,d'q = Sd'q\,f dq \quad \ldots \ldots \quad (357),$$

consequently the function f is self-conjugate, for dq and $d'q$ are quite arbitrary.

87. Differentiating (354) we find on comparison with (353)

$$dP = nSq\,dp, \qquad \text{where } (n-1)(m-1) = 1 \quad \ldots \ldots \quad (358),$$

and it is easy to verify that n is the order in which p is involved in P. Also introducing a new linear function g, we write

$$dq = (n-1)g\,dp \quad \ldots \ldots \ldots \quad (359),$$

and, as in the last article, g is self-conjugate and involves p in the order $n-2$ in its constitution.

Thus for any differential by (355) and (359)

$$dp = (m-1)f dq = (m-1)(n-1)fg\,dp = fg\,dp \quad \ldots \quad (360);$$

or symbolically

$$1 = fg = gf \quad \ldots \ldots \ldots \quad (361),$$

and one function produces on an arbitrary quaternion the same effect as the inverse of the other. In particular, employing EULER's theorem in (355) and (359) we have

$$p = fq = g^{-1}q; \quad q = gp = f^{-1}p \quad \ldots \ldots \ldots \quad (362).$$

88. When dq, instead of being perfectly arbitrary, satisfies

$$dQ = 0 \quad \text{or} \quad Sp\, dq = 0, \quad \text{where} \quad Q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (363),$$

dq represents some point in the tangent plane at q to the surface $Q = 0$. The point p is the reciprocal of the tangent plane with respect to the unit sphere $Sq^2 = 0$; and the surface $P = 0$ is the reciprocal of the given surface. The relations of reciprocity are clearly exhibited by the equations (compare (354))

$$Sp\, dq = 0, \quad Sq\, dp = 0, \quad dP = 0 \text{ if } Q = 0, \quad dQ = 0 \quad . \quad . \quad . \quad (364);$$

$$-S\, dpdq = Sp\, d^2q = Sq\, d^2p, \quad d^2P = 0 \text{ if also } d^2Q = 0 \quad . \quad . \quad (365).$$

89. For the asymptotic lines, in addition to (364) and (365), the new relation

$$0 = S\, dp\, dq = Sp\, d^2q = Sq\, d^2p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (366);$$

and thus for arbitrary scalars x and y

$$S\,(p + x\, dp)\,(q + y\, dq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (367),$$

or the reciprocal of an asymptotic tangent is the asymptotic tangent to the reciprocal surface at the corresponding point. Hence also, if corresponding tangents are reciprocal they touch asymptotic lines.

The tangents to the asymptotic lines of the original surface are also represented by the equations

$$Srfr = 0, \quad Spr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (368);$$

and those of the reciprocal surface by

$$Srgr = 0, \quad Sqr = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (369);$$

r being allowed to vary arbitrarily, but p and q being kept constant. These lines are, in fact, the generators of the reciprocal quadrics

$$Srfr = 0 \quad \text{or} \quad Srg^{-1}r = 0, \quad \text{and} \quad Srgr = 0 \quad \text{or} \quad Srf^{-1}r = 0 \quad . \quad (370),$$

(compare (362)) which lie in the corresponding tangent planes.

90. The generalized normal to a surface at any point is the line joining that point to the pole of the tangent plane with respect to the quadric of reciprocation. But as there is practically no additional labour involved in the following discussion when the auxiliary quadric is arbitrarily selected, we assume it to be

$$Sqhq = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (371);$$

and then the equation of the normal at q to the surface $Q = 0$ is

$$r = q + th^{-1}p \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (372).$$

If c is a centre of generalized curvature, or a point at which consecutive normals intersect, we have for intersecting normals

$$c = q + th^{-1}p, \quad \mathrm{d}c = \mathrm{d}q + th^{-1}\,\mathrm{d}p + h^{-1}p\,\mathrm{d}t = c\mathrm{d}u \quad . \quad . \quad . \quad (373),$$

where $\mathrm{d}u$ is some small scalar, and $\mathrm{d}c = c\mathrm{d}u$, because on the hypothesis that consecutive normals intersect in c, c and $\mathrm{d}c$ represent the same point and differ only in weight. On elimination of c, (373) becomes

$$\mathrm{d}q + th^{-1}\,\mathrm{d}p + v(q + th^{-1}p) + wq = 0, \quad (tv = \mathrm{d}t - t\,\mathrm{d}u,\ v + w = -\mathrm{d}u) \quad . \quad (374);$$

and as this may be written

$$(1 + th^{-1}f)(\mathrm{d}q + vq) + wq = 0, \quad \text{or} \quad \mathrm{d}q + vq + w(1 + th^{-1}f)^{-1}q = 0 \quad . \quad (375),$$

we find, on operating by $\mathrm{S}p$ or $\mathrm{S}fq$, the equation

$$\mathrm{S}qf(1 + th^{-1}f)^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (376).$$

On inversion of the function this becomes a quadratic in t whose roots determine the two centres of curvature.

91. This equation may be thrown into the more suggestive form*

$$\mathrm{S}q(f^{-1} + th^{-1})^{-1}q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (377),$$

which shows that the roots t are the parameters of two of the quadrics of the singly infinite system $\mathrm{S}r(f^{-1} + th^{-1})^{-1}r = 0$, which pass through the point q. The third quadric of the system through that point is of course $\mathrm{S}rfr = 0$, which corresponds to $t = 0$. The quadric $t = \infty$ is the auxiliary (371).

The two centres of curvature (373) are (t_1 and t_2 being the roots of (377))

$$c_1 = (f^{-1} + t_1h^{-1})p, \quad c_2 = (f^{-1} + t_2h^{-1})p \quad . \quad . \quad . \quad . \quad (378);$$

and the form of these equations shows that the points are the poles of the tangent plane $\mathrm{S}rp = 0$ with respect to the two quadrics t_1 and t_2.

The equation of the tangent to a line of curvature, $r = q + x\,\mathrm{d}q$ may by (375) be thrown into the form

$$r = q + yf^{-1}(f^{-1} + th^{-1})^{-1}q = q(1 + y) - yth^{-1}(f^{-1} + th^{-1})^{-1}q \quad . \quad (379),$$

where $t = t_1$ or t_2, and the form of this equation shows that the tangents are the generalized normals to the quadrics t_1 and t_2.

The first form of (379) shows that the tangent t_1 touches the quadric t_2, for

$$\mathrm{S}q(f^{-1} + t_2h^{-1})^{-1}f^{-1}(f^{-1} + t_1h^{-1})^{-1}q = 0 \quad . \quad . \quad . \quad . \quad (380),$$

as appears on replacing the middle function by

* Because $(1 + th^{-1}f)^{-1} = ((f^{-1} + th^{-1})f)^{-1} = f^{-1}(f^{-1} + th^{-1})^{-1}$.

$$(t_2 - t_1)f^{-1} = t_2(f^{-1} + t_1h^{-1}) - t_1(f^{-1} + t_2h^{-1}) \quad . \quad . \quad . \quad . \quad (381);$$

and, moreover, the lines of curvature form a conjugate *réseau* on the surface, for (380) gives

$$\mathrm{S}r_1fr_2 = 0 \text{ if } r_1 = f^{-1}(f^{-1} + t_1h^{-1})^{-1}q, \; r_2 = f^{-1}(f^{-1} + t_2h^{-1})^{-1}q \quad . \quad (382),$$

(compare (379)).

The other usual properties analogous to those for confocals may be easily obtained, but it must suffice to state that the centres of curvature for the quadric t_1 are

$$c' = f^{-1}(f^{-1} + t_1h^{-1})^{-1}q, \quad c_2' = (f^{-1} + t_2h^{-1})(f^{-1} + t_1h^{-1})^{-1}q \quad . \quad (383).$$

92. To reduce the equation (377) to a quadratic, let the symbolic quartic of $h^{-1}f$ be

$$(h^{-1}f)^4 - \mathrm{N}'''(h^{-1}f)^3 + \mathrm{N}''(h^{-1}f)^2 - \mathrm{N}'(h^{-1}f) + \mathrm{N} = 0 \quad . \quad . \quad (384);$$

then on multiplying by t^4 and dividing by $1 + th^{-1}f$, the result is

$$t^3\{(h^{-1}f)^3 - \mathrm{N}'''(h^{-1}f)^2 + \mathrm{N}''(h^{-1}f) - \mathrm{N}'\} - t^2\{(h^{-1}f)^2 - \mathrm{N}'''(h^{-1}f) + \mathrm{N}''\}$$
$$+ t\{(h^{-1}f) - \mathrm{N}'''\} - 1 = -\mathrm{N}_t(1 + th^{-1}f)^{-1} \quad . \quad . \quad . \quad . \quad (385).$$

Observing that the coefficient of t^3 on the left is $-\mathrm{N}(h^{-1}f)^{-1}$ or $-\mathrm{N}f^{-1}h$, the equation (376) becomes

$$t^3\mathrm{NS}qhq + t^2\mathrm{S}qf\{(h^{-1}f)^2 - \mathrm{N}'''(h^{-1}f) + \mathrm{N}''\}q$$
$$- t\mathrm{S}fq\{h^{-1}f - \mathrm{N}'''\}q + \mathrm{S}qfq = 0 \quad . \quad . \quad . \quad . \quad (386);$$

and this immediately reduces to

$$t^2\mathrm{NS}qhq + t\mathrm{S}p(h^{-1}fh^{-1} - \mathrm{N}'''h^{-1})p + \mathrm{S}ph^{-1}p = 0 \quad . \quad . \quad . \quad (387),$$

when we replace fq by p, and discard the extraneous factor t.

If n and n_1 are the fourth invariants of f and h, $\mathrm{N} = nn_1^{-1}$; and it is easy to see that n is the result of substituting q in the equation of the Hessian of the surface if Q is an integral as well as a homogeneous function of q. Thus one root is infinite in either of two cases, if the point is on the Hessian, and if it is on the auxiliary quadric; in either case the centre of curvature is the pole of the tangent plane with respect to the auxiliary. A root is zero if $\mathrm{S}ph^{-1}p = 0$, and in this case the tangent plane touches the auxiliary, and a centre of curvature is the point q itself. These special cases depend on two distinct conditions, the relation of the auxiliary quadric to the surface, and the relation of the Hessian to the surface.

93. A curve is a generalized geodesic when consecutive tangents are coplanar with the pole of the tangent plane with respect to the auxiliary quadric; or, symbolically,

$$(q, \mathrm{d}q, \mathrm{d}^2q, h^{-1}p) = 0, \quad \text{or} \quad xq + y\,\mathrm{d}q + z\,\mathrm{d}^2q + wh^{-1}p = 0 \quad . \quad . \quad (388)$$

is the equation of a geodesic.

Operate with $\mathrm{S}p$, $\mathrm{Sd}p$, $\mathrm{S}hq$, $\mathrm{S}h\mathrm{d}q$ and by (364), (365),

$$z\mathrm{S}p\mathrm{d}^2q + w\mathrm{S}ph^{-1}p = 0\,; \quad y\mathrm{Sd}p\,\mathrm{d}q + z\mathrm{Sd}p\,\mathrm{d}^2q + w\mathrm{Sd}ph^{-1}p = 0\,;$$
$$x\mathrm{S}qhq + y\mathrm{S}qh\,\mathrm{d}q + z\mathrm{S}qh\,\mathrm{d}^2q = 0\,; \quad x\mathrm{S}qh\,\mathrm{d}q + y\mathrm{Sd}qh\,\mathrm{d}q + z\mathrm{Sd}qh\,\mathrm{d}^2q = 0 \quad . \quad (389).$$

Introducing the function f and eliminating the scalars $xyzw$, we find

$$\frac{\mathrm{Sd}ph^{-1}p}{\mathrm{S}ph^{-1}p} = -\frac{y\mathrm{Sd}qf\,\mathrm{d}q + z\mathrm{Sd}qf\,\mathrm{d}^2q}{z\mathrm{Sd}qf\,\mathrm{d}q}$$
$$= -\frac{\mathrm{Sd}qf\,\mathrm{d}^2q}{\mathrm{Sd}qf\,\mathrm{d}q} + \frac{\mathrm{S}qhq\mathrm{Sd}qh\,\mathrm{d}^2q - \mathrm{S}qh\,\mathrm{d}q\mathrm{S}qh\,\mathrm{d}^2q}{\mathrm{S}qhq\mathrm{Sd}qh\,\mathrm{d}q - (\mathrm{S}qh\,\mathrm{d}q)^2} \quad . \quad . \quad . \quad (390)\,;$$

and this, when the surface is a quadric so that f is constant, immediately integrates, and gives

$$\mathrm{S}ph^{-1}p\mathrm{Sd}qf\,\mathrm{d}q = u\,(\mathrm{S}qhq\mathrm{Sd}qh\,\mathrm{d}q - (\mathrm{S}qh\,\mathrm{d}q)^2) \quad . \quad . \quad . \quad . \quad (391),$$

where u is the constant of integration.

SECTION XV.

The Analogue of Hamilton's Operator ∇.

94. In applications of quaternions to projective geometry an operator analogous to Hamilton's ∇ is occasionally useful. I define it by the equation (compare Art. 85)

$$\mathrm{D}Q = p \quad \text{when} \quad \mathrm{d}Q = \mathrm{S}p\mathrm{d}q \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (392).$$

To render this operator available for use, take any four independent differentials of q and write down the identity

$$p\,(\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q) = [\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q]\,\mathrm{S}p\,\mathrm{d}q - [\mathrm{d}q\,\mathrm{d}''q\,\mathrm{d}'''q]\,\mathrm{S}p\,\mathrm{d}'q$$
$$+ [\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}'''q]\,\mathrm{S}p\,\mathrm{d}''q - [\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}''q]\,\mathrm{S}p\,\mathrm{d}'''q \quad . \quad (393),$$

which suggests the symbolical equation

$$\mathrm{D} = \Sigma\,\frac{\pm\,[\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q]\,\mathrm{d}}{(\mathrm{d}q\,\mathrm{d}'q\,\mathrm{d}''q\,\mathrm{d}'''q)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (394),$$

where the summation refers to the four symbols d.

95. Otherwise, if the quaternion variable q is a function of four parameters, x, y, z, w, we may replace the arbitrary differentials in terms of the deriveds of q with respect to these parameters, and then (394) becomes

$$\mathrm{D} = \Sigma \frac{\pm [q_y q_z q_w]}{(q_x q_y q_z q_w)} \frac{\partial}{\partial x} \quad \ldots \quad (395),$$

where

$$q_x = \frac{\partial q}{\partial x}, \quad q_y = \frac{\partial q}{\partial y}, \quad q_z = \frac{\partial q}{\partial z}, \quad q_w = \frac{\partial q}{\partial w} \quad \ldots \quad (396).$$

In particular, if these four deriveds satisfy the six equations

$$\mathrm{S}q_x q_y = \mathrm{S}q_y q_z = \mathrm{S}q_z q_x = \mathrm{S}q_x q_w = \mathrm{S}q_y q_w = \mathrm{S}q_z q_w = 0 \quad \ldots \quad (397),$$

it easily appears that the symbolic equation (395) reduces to

$$\mathrm{D} = \frac{q_x}{\mathrm{S}q_x^2} \frac{\partial}{\partial x} + \frac{q_y}{\mathrm{S}q_y^2} \frac{\partial}{\partial y} + \frac{q_z}{\mathrm{S}q_z^2} \frac{\partial}{\partial z} + \frac{q_w}{\mathrm{S}q_w^2} \frac{\partial}{\partial w} \quad \ldots \quad (398).$$

More particularly if q is referred to the vertices of a tetrahedron self-conjugate to the unit sphere, so that

$$q = ax + by + cz + dw, \quad \text{and if} \quad \mathrm{S}a^2 = \mathrm{S}b^2 = \mathrm{S}c^2 = \mathrm{S}d^2 = 1 \quad . \quad (399)$$

for suitable selection of the weights of these four points, the operator takes its simplest form

$$\mathrm{D} = a \frac{\partial}{\partial x} + b \frac{\partial}{\partial y} + c \frac{\partial}{\partial z} + d \frac{\partial}{\partial w} \quad \ldots \quad (400),$$

while

$$\mathrm{SD}^2 = \left(\frac{\partial}{\partial x}\right)^2 + \left(\frac{\partial}{\partial y}\right)^2 + \left(\frac{\partial}{\partial z}\right)^2 + \left(\frac{\partial}{\partial w}\right)^2 \quad \ldots \quad (401).$$

If, on the other hand, $\quad q = t + ix + jy + kz \quad \ldots \quad (402),$

the operator reduces to $\quad \mathrm{D} = \dfrac{\partial}{\partial t} - \nabla \quad \ldots \quad (403).$

96. It may be useful to collect a few formulæ which may serve as examples of the application of the operator. We therefore give the following:

$$\mathrm{D}q = 4\,;\ \mathrm{DK}q = -2 = \mathrm{KD}q\,;\ \mathrm{DS}q = 1 = \mathrm{SD}q\,;\ \mathrm{DV}q = 3 = \mathrm{VD}q\,;$$
$$\mathrm{DS}aq = a\,;\ \mathrm{DS}.q^2 = 2q\,;\ \mathrm{DT}q^2 = 2\mathrm{K}q\,;\ \mathrm{D}q^2 = 4(q + \mathrm{S}q)\,;\ \mathrm{D}(\mathrm{V}q)^2 = 2\mathrm{V}q\,;$$
$$\mathrm{DT}(q + a) = \mathrm{KU}(q + a)\,;\ \mathrm{DS}qfq = (f + f')\,q.$$

To these we may add

$$\mathrm{D}^2\mathrm{T}(q + a)^2 = -4 = \mathrm{TD}^2\mathrm{S}(q + a)^2\,;\ \mathrm{TD}^2\mathrm{T}(q + a)^2 = 8 = \mathrm{D}^2\mathrm{S}(q + a)^2\,;$$
$$\mathrm{TD}^2.\mathrm{T}q^n = n\mathrm{KD}.\mathrm{K}q\mathrm{T}q^{n-2} = n(4\mathrm{T}q^{n-2} + (n-2)\,q\mathrm{K}q\mathrm{T}q^{n-4}) = n(n+2)\,\mathrm{T}q^{n-2}.$$

And again

$$\mathrm{D}^2(\mathrm{S}.q^2)^n = 2n\mathrm{D}.q(\mathrm{S}.q^2)^{n-1} = 8n(\mathrm{S}.q^2)^{n-1} + 4n(n-1)\,q^2(\mathrm{S}.q^2)^{n-2}\,;$$

and on taking the scalar of both sides

$$S.D^2.(S.q^2)^n = 4n(n+1)(S.q^2)^{n-1}.$$

From these results follow certain analogues of LAPLACE'S equation

$$TD^2Tq^{-2} = 0, \quad TD^2.f(D).T(q+a)^{-2} = 0 \quad . \quad . \quad . \quad . \quad (404);$$

and

$$S.D^2(S.q^2)^{-1} = 0, \quad S.D^2.f(D).(S.(q+a)^2)^{-1} = 0 \quad . \quad . \quad . \quad (405).$$

Moreover, the general expression for the operator in terms of arbitrary differentials a, b, c, d of q enables us to write down a number of invariants and identities. For instance, operating on fq, we find

$$D.fq.(abcd) = [bcd]fa - [acd]fb + [abd]fc - [abc]fd \quad . \quad . \quad (406).$$

Other examples relating to integration will be found in a paper in 'Proc. Roy. Irish Acad.,' vol. 24, Sect. A, pp. 6–20.

97. So far as projective geometry is concerned, the use we make of the operator D is to form successive polars of a point with respect to a surface and to show that it leads directly to ARONHOLD'S notation.

The n^{th} polar of a point r with respect to a surface $Q = 0$ of order m is

$$(SrD)^n.Q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (407).$$

If $n = m$, the operator simply multiplies Q by a numerical factor and changes the quaternion involved from q to r. Thus we may write the equation of the surface in the form

$$(SrD)^mQ = 0, \quad \text{or} \quad (Sra)^m = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (408),$$

where a is a symbolic quaternion devoid of meaning unless it enters into a term homogeneous in a to the order m. This is equivalent to ARONHOLD'S method.

SECTION XVI.

THE BILINEAR QUATERNION FUNCTION.

98. We shall now explain a method which promises to be of considerable value in the application of quaternions to projective geometry.

A bi-linear quaternion function $f(pq)$ is a function of two quaternions (p and q) linear and distributive with respect to both. It may be reduced to the form

$$f(pq) = a_1 \mathrm{S}pf_1q + a_2\mathrm{S}pf_2q + a_3\mathrm{S}pf_3q + a_4\mathrm{S}pf_4q \quad . \quad . \quad . \quad . \quad (409),$$

where a_1, a_2, a_3, a_4 are any four quaternions and where f_1, f_2, f_3, and f_4 are four linear quaternion functions. The bilinear function involves sixty-four constants, sixteen for each of the four functions.

99. Writing generally for all quaternions p and q

$$f(pq) = f_{\prime}(qp) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (410),$$

we may call the new bilinear function $f_{\prime}$ the *permutate* of the function f. When a function is unaltered by transposition of the quaternions, it may be called a *permutable* function. Thus

$$\mathrm{P}(pq) = \tfrac{1}{2}f(pq) + \tfrac{1}{2}f_{\prime}(pq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (411)$$

is a permutable function, the permutable part of f or $f_{\prime}$. A permutable function involves forty constants, the functions f_1, f_2, f_3, f_4 of (409) being then self-conjugate.

100. When a bilinear function changes sign with transposition of its quaternions, it may be called a *combinatorial* function. Thus

$$\mathrm{C}(pq) = \tfrac{1}{2}f(pq) - \tfrac{1}{2}f_{\prime}(pq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (412)$$

is combinatorial. It vanishes for $p = q$, and, regarded geometrically, it relates not to a pair of points, but to the line joining the points.

A bilinear function is thus reducible to the form

$$f(pq) = \mathrm{P}(pq) + \mathrm{C}(pq)\,; \quad f_{\prime}(pq) = \mathrm{P}(pq) - \mathrm{C}(pq) \quad . \quad . \quad (413)\,;$$

and is uniquely resoluble into its permutable and combinatorial parts.

101. Writing generally for any three quaternions, p, q, and r,

$$\mathrm{S}rf(pq) = \mathrm{S}pf'(rq) = \mathrm{S}qf''(pr) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (414),$$

we shall call the new functions $f'(pq)$, $f''(pq)$ the *first and second conjugates* of $f(pq)$. In fact $f'(pq)$ is the conjugate when the first quaternion p alone varies, and $f''(pq)$ is the conjugate when the second varies.

102. As the accents employed to denote the permutate and the first and second conjugates are not commutative in order of application, it is safer to use brackets in the rare cases in which double accents are necessary. Thus

$$f(pq) = (f')'(pq) = (f'')''(pq) = (f_{\prime})_{\prime}(pq) \quad . \quad . \quad . \quad . \quad (415),$$

because the first conjugate of the first conjugate of $f(pq)$ is simply the function $f(pq)$ itself.

When the successive accents are different, the laws connecting the various functions are deducible from the relations (compare (414))

$$\begin{aligned} Srf(pq) &= Spf'(rq) = Sq(f')''(rp) = Sp(f')_{,}(qr) \\ &= Sqf''(pr) = Sq(f'')_{,}(rp) = Sp(f'')'(qr) \\ &= Srf_{,}(qp) = Sq(f_{,})'(rp) = Sp(f_{,})''(qr) \quad . \quad . \quad . \quad (416), \end{aligned}$$

in which p, q and r are perfectly arbitrary.

These relations show that

$$\begin{aligned} (f')''(pq) &= (f'')_{,}(pq) = (f_{,})'(pq) = f''(qp), \\ (f')_{,}(pq) &= (f'')'(pq) = (f_{,})''(pq) = f'(qp) \quad . \quad . \quad . \quad . \quad (417); \end{aligned}$$

and thus any multiply accented function may be reduced to one or other of six fundamental functions, the function and its two conjugates and the permutates of these three functions.

103. Exactly as in Arts. 5 and 6, the equations

$$\begin{aligned} &(f(aq) - ta\,;\, f(bq) - tb\,;\, f(cq) - tc\,;\, f(dq) - td) \\ &\qquad = (f'(aq) - ta\,;\, f'(bq) - tb\,;\, f'(cq) - tc\,;\, f'(dq) - td) \quad (418), \end{aligned}$$

$$\begin{aligned} &(f(pa) - ta\,;\, f(pb) - tb\,;\, f(pc) - tc\,;\, f(pd) - td) \\ &\qquad = (f''(pa) - ta\,;\, f''(pb) - tb\,;\, f''(pc) - tc\,;\, f''(pd) - td) \quad (419) \end{aligned}$$

are identities for all quaternions p, q, a, b, c and d, and for every value of the scalar t. The first is obtained on the supposition that $f(pq)$ is a function of p, and the second on the supposition that it is a function of q. Dividing each member of the identities by $(abcd)$, we obtain the biquadratics

$$\begin{aligned} &J(q) - tJ'(q) + t^2J''(q) - tJ'''(q) + t^4, \\ &I(p) - tI'(p) + t^2I''(p) - tI'''(p) + t^4 \quad . \quad . \quad . \quad . \quad . \quad (420); \end{aligned}$$

and $J(q)$, $J'(q)$, $J''(q)$, $J'''(q)$, of the fourth, third, second and first order respectively in q, are the invariants of $f(pq)$ considered as a function of p. Equating these biquadratics to zero, we obtain the equations whose roots are the latent roots of $f(pq)$ as a function of p and as a function of q.

It is evident from (418) and (419) that these relations are equivalent when the function is permutable, and then $I(q) = J(q)$, &c.

104. The quartic surfaces

$$I(p) = 0, \quad J(q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (421)$$

we shall call respectively the first and second Jacobians. Whenever a pair of quaternions satisfies the equation

$$f(pq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (422),$$

the point p must lie on the surface $I(p) = 0$ and q must lie on $J(q) = 0$; for $f(pr)$, a linear function of r, has then one zero latent root, and $f(rq)$ has also a zero latent root.

On reference to (409), it appears that (422) is equivalent to

$$Spf_1q = Spf_2q = Spf_3q = Spf_4q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (423);$$

and in the particular case when the function is permutable, the four linear functions are self-conjugate, and the equations assert that the polar planes of one point (p) intersect in the other (q). In this case the surfaces (421) coincide with one another and with the Jacobian of the four quadrics; and although it does not appear that in general the surfaces are the Jacobians of four quadrics, we have retained the name as being convenient and suggestive.

Two points related as in (422) will be called *Jacobian correspondents*, or more particularly IJ Jacobian correspondents.

105. When a function has a zero latent root, so has its conjugate. Consequently, whenever p and q are Jacobian correspondents, or whenever (422) is satisfied, it must be possible to find two other points r' and r'', so that

$$f'(r'q) = 0, \quad f''(pr'') = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (424).$$

There are thus two new types of Jacobian correspondence; and the argument of Art. 102 shows that there can be no more, for the conditions (422) and (424) may be re-written in the form

$$f_{\prime}(qp) = 0, \quad (f')_{\prime}(qr') = 0, \quad (f'')_{\prime}(r''p) = 0 \quad . \quad . \quad . \quad . \quad (425),$$

without altering the signification of the equations, and we have now exhausted the six fundamental functions of the article cited.

106. *The points "r'" and "r''" of the second and third Jacobian correspondences lie upon the third Jacobian $K(r)$.*

A latent root of $f'(r'q)$ considered as a function of q (424) is zero, and therefore r' satisfies the equation

$$(f'(r'a), f'(r'b), f'(r'c), f'(r'd)) = ((f')''(r'a), (f')''(r'b), (f')''(r'c), (f')''(r'd)) = 0 \;.\; (426),$$

in the second number of which the function of q has been replaced by its conjugate. But (417) the second number is equivalent to

$$(f''(ar), f''(br), f''(cr), f''(dr)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (427),$$

and consequently r'', which satisfies (427), satisfies also (426), or r' and r'' lie upon the same quartic surface.

As in Art. 103, we deduce the identity

$$(f'(ra) - ta\,;\; f'(rb) - tb\,;\; f'(rc) - tc\,;\; f'(rd) - td)$$
$$= (f''(ar) - ta\,;\; f''(br) - tb\,;\; f''(cr) - tc\,;\; f''(dr) - td) = 0 \quad . \quad (428);$$

and the result of dividing by $(abcd)$ may be written in the form

$$K(r) - tK'(r) + t^2K''(r) - t^3K'''(r) + t^4 \quad . \quad . \quad . \quad . \quad . \quad . \quad (429),$$

and the latent quartic of $f'(rq)$ or $f''(qr)$ (functions of q) is obtained by equating this to zero.

The scheme of the Jacobians is now complete, the six fundamental functions of Art. 102 having been employed.

The points $r'q$ of (424) may be said to be JK Jacobian correspondents, and p and r'' are IK correspondents.

When $f(pq)$ is permutative, the JK and IK types unite and I coincides with J; when $f(pq)$ is self-conjugate with respect to p, K coincides with I, and the JK and IJ correspondences coalesce.

It readily appears from (416) that when the function is doubly self-conjugate it is also permutable, and when it is permutable and self-conjugate to one element it is likewise self-conjugate to the other. In this case the three Jacobians coincide with the Hessian of the cubic surface

$$Sqf(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (430).$$

SECTION XVII.

The Four-System of Linear Functions.

An Example of the Use of the Bilinear Function.

107. When one of the quaternions in a bilinear function is regarded as a quaternion parameter, the function represents a triply-infinite system of linear quaternion functions, or a *four-system* of linear functions, to borrow a convenient phrase from Sir ROBERT BALL'S 'Theory of Screws.'

Thus

$$f(pq) = x_1 f(p_1 q) + x_2 f(p_2 q) + x_3 f(p_3 q) + x_4 f(p_4 q),$$
$$\text{where} \quad p = x_1 p_1 + x_2 p_2 + x_3 p_3 + x_4 p_4 \quad . \quad . \quad (431)$$

is a linear combination of four given linear functions $f(p_{\prime\prime} q)$, the quaternions $p_{\prime\prime}$ being supposed given while the scalars $x_{\prime\prime}$ are variable.

It is frequently of advantage to use the notation

$$f(pq) = f_p(q) = f_q(p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (432),$$

when the bilinear function is regarded as a function of q or as a function of p.

108. *An arbitrary point is a united point of a definite function of the four-system, provided it does not lie on a critical curve of the tenth order.*

If q is assumed to be a united point of a function determined by p,

$$\{f(pq), q\} = 0, \quad \text{or} \quad f(pq) = tq, \quad \text{or} \quad f_q(p) = tq \quad . \quad . \quad . \quad (433);$$

and the solution of the equation in its third form is

$$pJ(q) = tF_q(q), \quad \text{or} \quad p = F_q(q), \; t = J(q) \quad . \quad . \quad . \quad . \quad (434),$$

where F_q is HAMILTON'S auxiliary function corresponding to f_q and where $J(q)$ is the fourth invariant of f_q (Art. (103)).

This solution is definite (Art. 15), provided q does not lie upon the critical curve

$$F_q(q) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (435).$$

To exhibit the nature of this curve, observe that

$$0 = \mathrm{S} \,.\, F_q(q)\,[p_1p_2p_3] = \mathrm{S}qF_q'\,[p_1p_2p_3] = [q,\, f_qp_1,\, f_qp_2,\, f_qp_3] \,. \quad . \quad (436)$$

for all quaternions $[p_1p_2p_3]$, $[p_1p_2p_4]$, &c.; or, in the notation of Art. 65,

$$((q,\, f(p_1q),\, f(p_2q),\, f(p_3q),\, f(p_4q))) = 0 \,. \quad . \quad . \quad . \quad . \quad (437),$$

whenever (435) is satisfied. But we have seen that (437) represents a curve of order $m = 10$ and rank $r = 40$ (278), which is common to all the quartic surfaces obtained by deleting one quaternion within the double brackets (436).

The solution may be expressed in a more explicit form by means of the identity

$$q(f(p_1q), f(p_2q), f(p_3q), f(p_4q)) = \Sigma \pm f(p_1q)\,(q, f(p_2q), f(p_3q), f(p_4q)) \quad . \quad (438),$$

so that we may write (434) in the form

$$p\,(p_1p_2p_3p_4) = \Sigma \pm p_1\,(q,\, f(p_2q),\, f(p_3q),\, f(p_4q))\,; \quad t = J(q) \quad . \quad (439).$$

109. *When the point lies on the critical curve it is generally a united point of every function of a determinate two-system.*

In this case the solution of (433) is (Art. 15)

$$pJ'(q) = tG_q(q) + F_q(p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (440)\,;$$

or

$$p = G_q(q) + xp_0, \quad f(p_0q) = 0, \quad t = J'(q) \quad . \quad . \quad . \quad . \quad (441).$$

Thus p may be any point on the line joining the point $G_q(q)$ to p_0 — the Jacobian correspondent of q; and consequently a determinate two-system exists, every function of which has q for a united point (compare Art. 123).

110. Similarly for the conjugate four-system $f''(pr)$, a point r is a united point of a definite function, unless it happens to lie upon the conjugate critical curve

$$F_r^{\backprime}(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (442),$$

where $F_r^{\backprime}$ is the auxiliary function of $f_r^{\backprime}(p) = f''(pr)$, but we must observe that $f_q^{\backprime}$ is not the conjugate of f_q.

Now the reciprocal of a united point of $f''(pr)$ (the conjugate to r of $f(pr)$) is a united plane of the original four-system. And thus an arbitrary plane is the united plane of some definite function, but if the plane belongs to the *developable surface* (442) it is a common united plane of a definite two-system of functions determined by

$$p = G_r^{\backprime}(r) + xp_0^{\backprime}, \quad f''(p_0^{\backprime}r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (443).$$

Ten of these singular planes pass through an arbitrary point; the order of the developable surface is $r = 40$; and the order of the cuspidal curve* is $n = 3(r - m) + \beta = 90$.

* 'Three Dimensions,' Art. 327.

111. It is obvious from this theory that the united points of functions of this system compose definite tetrads, so that one point of a tetrad being given the remaining three are generally determinate.

In fact (434) is a quartic transformation connecting united points q with the auxiliary points p, so that one point p corresponds to one point q, while four points q correspond to one point p. For a given point p, these four points are by (434) the intersections of the quartic surfaces, for arbitrary quaternions l,

$$\frac{Sl_1F_q(q)}{Sl_1p} = \frac{Sl_2F_q(q)}{Sl_2p} = \frac{Sl_3F_q(q)}{Sl_3p} = \frac{Sl_4F_q(q)}{Sl_4p} \quad . \quad . \quad . \quad . \quad . \quad (444).$$

But these surfaces have a common curve (435); and three surfaces having a common curve intersect in

$$\mu\nu\rho - m(\mu + \nu + \rho - 2) + r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (445)$$

points not on the common curve, and this number is 4 when $\mu = \nu = \rho = 4$, $m = 10$, $r = 40$, as in the present case.

112. *The locus of points "p" determining functions, each of which has a united point on a given line, is a unicursal twisted quartic.*

When we replace q by $q + xq'$ in the second form of (434), we may write

$$p = (p_0p_1p_2p_3p_4 \between x, 1)^4 = p_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (446),$$

and the form of the equation establishes the proposition.

In like manner we have

$$t = (t_0t_1t_2t_3t_4 \between x, 1)^4 = t_x \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (447).$$

113. *For every intersection of the line with the critical curve, the quartic breaks up.*

If x' is the value of the scalar x for a point on the critical curve, $p_{x'}$ and $t_{x'}$ both vanish, or

$$0 = (p_0p_1p_2p_3p_4 \between x', 1)^4, \qquad 0 = (t_0t_1t_2t_3t_4 \between x', 1)^4 \quad . \quad . \quad . \quad . \quad (448).$$

We may employ these equations to eliminate p_4 and t_4 from (446) and (447); and discarding the factor $x - x'$, we find

$$p = (p'_0p'_1p'_2p'_3 \between x1)^3, \qquad t = (t'_0t'_1t'_2t'_3 \between x1)^3 \quad . \quad . \quad . \quad . \quad (449).$$

The locus of p is now a twisted cubic, and the discarded factor corresponds to a line of the nature of those of Art. 109.

When the line qq' meets the critical curve twice, the locus is a conic and a pair of lines. If the line is a triple chord, the locus is one line of a new type and three lines of the type already mentioned. Finally, for a quadruple chord, the quartic reduces to a point and four lines, as we shall see immediately.

But first we notice that the arguments of Art. 110 apply, so that we may write down the equation of the quartic curve whose points determine functions, each of

which has a united plane through a given line. If the line lies in one or more of the planes of the developable (442), the quartic degrades in the manner explained.

114. Otherwise we may say that (446) and (447) determine a system of functions $f(p_x q) - t_x q$ which destroys the line $q + xq'$ point by point. Or counting unity as one function, it may be said that a five-system is required to destroy a line point by point. However, when the line intersects the critical curve once, twice, or thrice, it can be destroyed seriatim by a four-, three-, or two-system of functions. For example, in the case of triple intersection we may write

$$p_x = p_0 x + p_1,\ t_x = t_0 x + t_1;\ f(p_0 x + p_1,\ q'x + q) - (t_0 x + t_1)(q'x + q) = 0 \quad (450);$$

and, going one step further, in the case of a quadruple chord

$$f(p_0,\ q'x + q) = t_0 (q'x + q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (451).$$

Thus a quadruple chord of the critical curve is a line locus of united points of a determinate function. And because the number of quadruple chords of a curve is ('Three Dimensions,' Art. 274)

$$\tfrac{1}{24}(-m^4 + 18m^3 - 71m^2 + 78m - 48mh + 132h + 12h^2) \quad . \quad . \quad (452),$$

or 20 for $m = 10$, $h = 25$, we learn that *twenty functions of the four-system have line loci of united points—quadruple chords of the critical curve.*

The formula (314) gives 80 as the order of the surface of triple chords.

115. *The locus of a point which determines a function having a united point in a given plane is a sextic surface.*

The functions H_p, G_p and F_p being HAMILTON's auxiliary functions for $f_p(q) = f(pq)$, the relations

$$H_p(q) = t'q;\quad G_p(q) = t''q;\quad F_p(q) = t'''q \quad . \quad . \quad . \quad . \quad . \quad (453)$$

are satisfied, provided q is a united point of $f(pq)$, t', t'' and t''' being suitable scalars.

If q lies in a given plane, these equations, with that of the given plane, afford the relations

$$Sql = 0,\quad SqH_p'(l) = 0,\quad SqG_p'(l) = 0,\quad SqF_p'(l) = 0 \quad . \quad . \quad (454),$$

linear in q and of orders 0, 1, 2 and 3 in p. Expressing that q is a common point, we have the equation of the sextic surface

$$(l,\quad H_p'(l),\quad G_p'(l),\quad F_p'(l)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (455).$$

116. *The sextic surface has a double curve of the seventh order answering to pairs of united points in the plane.*

If the first, second and third of equations (454) regarded as planes in q intersect in a common line, the fourth plane will also pass through that line. The condition for a common line is

$$ul + vH_p'(l) + wG_p'(l) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (456),$$

where u, v and w are certain scalars. Operating on this by f_p', we have by Art. 6

$$u\,(I'''(p) - H_p')\,l + v\,(I''(p) - G_p')\,l + w\,(I'(p) - F_p')\,l = 0 \quad . \quad . \quad (457),$$

remembering (Art. 103) that $I'''(p)$, $I''(p)$, $I'(p)$ and $I(p)$ are the invariants of f_p'. But this relation gives $F_p'(l)$ linearly in terms of l, $H_p'(l)$, $G_p'(l)$, and therefore, as asserted, the fourth plane will also pass through the common line.

Hence it appears that (456), or its equivalent

$$[l,\ H_p'(l),\ G_p'(l)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (458),$$

represents a double curve on the sextic (455); for if p is any point on this curve, not only will (455) be satisfied, but the equation of the tangent plane at that point will also vanish, since every set of three quaternions included in the brackets of (455) is then linearly connected. The order of this curve is 7, by Art. 64.

Moreover, (456) expresses that a united line of the function f_p' passes through the point l, or, reciprocally, that a united line of the function f_p lies in the plane $Slq = 0$.

117. *The point determining the function for which the plane is a united plane is a triple point on the sextic.*

If p_0 is this point, and if t_1, t_2, t_3 are the roots of the function $f(p_0 q)$ answering to the united points in the plane, it follows from the fundamental properties of the auxiliary functions that

$$H_{p'_0}(l) = \Sigma t_1 \,.\, l, \quad G_{p'_0}(l) = \Sigma t_2 t_3 \,.\, l, \quad F_{p'_0}(l) = t_1 t_2 t_3 \,.\, l \quad . \quad . \quad (459);$$

and consequently the tangent plane and the polar quadric of the point p_0 to the surface (455) vanish identically. The point is therefore a triple point.

118. It may be noticed that in terms of a, b, c, any three points in the plane, the triple point is

$$p_0 = [f'(la),\ \ f'(lb),\ \ f'(lc)] \,.\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (460);$$

also in terms of these three points, if $l = [abc]$,

$$H_p'(l) = \Sigma\,[f(pa), b, c,], \quad G_p'(l) = \Sigma\,[a, f(pb), f(pc)],$$
$$F_p'(l) = [f(pa), f(pb), f(pc)] \,.\quad . \quad . \quad . \quad . \quad . \quad . \quad (461).$$

Consequently if $q = xa + yb + zc$, we may replace the system of equations (454) by

$$xX + yY + zZ = 0, \quad xX_2 + yY_2 + zZ_2 = 0, \quad xX_3 + yY_3 + zZ_3 = 0 \,. \quad (462),$$

where

$$X = SaH_p'(l) = (a, f(pa), b, c);$$
$$X_2 = SaG_p'(l) = (a, f(pa), f(pb), c) + (a, f(pa), b, f(pc));$$
$$X_3 = SaF_p'(l) = (a, f(pa), f(pb), f(pc)) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (463);$$

and Y, Y_2, Y_3 and Z, Z_2, Z_3 may be written down from symmetry.

Moreover, when we specially select the points a, b, c as the united points of the function $f(p_0q)$, and when we form successive polars of p_0 with respect to X, X_2 and X_3, we find (Art. 97) in terms of the latent roots t_1, t_2, t_3 corresponding to a, b and c,

$$\mathrm{S}p_0\mathrm{D}\,.\,X = 0,\quad \mathrm{S}p_0\mathrm{D}\,.\,X_2 = (t_2 + t_3)\,X,\quad (\mathrm{S}p_0\mathrm{D})^2\,X_3 = 2t_2tX_3 \quad . \quad (464),$$

because

$$\mathrm{S}p_0\mathrm{D}\,.\,Y_3 = (a, f(p_0a), f(pb), f(pc)) + (a, f(pa), f(p_0b), f(pc)) + (a, f(pa), f(pb), f(p_0c))$$
$$= t_2\,(a, f(pa), b, f(pc)) + t_3\,(a, f(pa), f(pb), c)\ .\ .\ .\ .\ .\ .\ .\ .\ \ (465),$$

and similarly in the other cases.

Thus the equation of the sextic may be written in the form

$$\begin{vmatrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} = 0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (466),$$

and the third polar of the point p_0 is

$$(t_2 - t_3)\,(t_3 - t_1)\,(t_1 - t_2)\,XYZ = 0\ .\ .\ .\ .\ .\ .\ .\ (467).$$

Thus the tangent cone at the triple point breaks up into three planes.

In the same notation the double curve is represented by

$$\left\| \begin{matrix} X & Y & Z \\ X_2 & Y_2 & Z_2 \end{matrix} \right\| = 0\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (468);$$

and forming the polars, the point p_0 is seen to be triple and

$$\left\| \begin{matrix} X & Y & Z \\ (t_2 + t_3)X, & (t_3 + t_1)Y, & (t_1 + t_2)Z \end{matrix} \right\| = 0\ .\ .\ .\ .\ (469)$$

represents the system of tangents at the triple points—the lines of intersection of the planes X, Y and Z.

We may add that the equation of the cone, vertex p_0, standing on the curve is

$$(t_2 - t_3)\,XYZ_2 + (t_3 - t_1)\,XYZ_2 + (t_1 - t_2)\,XYZ_2 = 0\ \ .\ .\ .\ (470).$$

119. This surface resembles a STEINER'S quartic in many particulars, but it is a degraded case of the general surface

$$p = (xyz)^4\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (471),$$

where $(xyz)^4$ is the general quaternion function of three homogeneous scalar parameters x, y, z. The general surface is of the 16th order. The STEINER quartic may be written $p = (xyz)^2$, a general quaternion quadratic function of x, y, z. Surfaces of this type arise from the general transformation

$$p = f(q, q, \ldots q)\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ (472)$$

of the nth order, being the transformations of planes.

The twisted quartics of Art. 112 correspond to the conics on the STEINER quartic. The sextic surface contains ten lines corresponding to the ten points in which the plane intersects the critical curve of the tenth order, for to every point on that curve corresponds a two-system of functions or a line in the space p (Art. 109). Again, the sextic contains an infinite number of twisted cubics corresponding to the lines in the plane which pass through one of these ten points (Art. 113); and it likewise contains 45 conics answering to the connectors of these points. More generally (Art. 113) a conic through five of these points transforms into a twisted cubic, and similarly for other cases.

120. When we express that the twisted cubic (449) is plane, the condition

$$(p'_0, p'_1, p'_2, p'_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (473)$$

is of the tenth order in q' and of the sixth in q, which latter point we may suppose to be on the critical curve. This condition will then represent a cone of the tenth order of the lines through the point q which transform into plane curves in the p space. But this cone must consist in part of the cone of the ninth order containing the critical curve. The remaining part is a plane, and every line in this plane through q transforms into a plane cubic.

In particular, an arbitrary plane cuts the critical curve in ten points and intersects ten planes of the type just mentioned in lines which transform into plane cubics on the sextic surface. Here again is a point of similarity with the STEINER quartic, for the plane containing one of these cubics cuts the sextic again in another cubic.

121. Corresponding to a plane $[p_1p_2p_3]$ in the p space there is a Jacobian quartic

$$(q, f(p_1q), \ f(p_2q), \ f(p_3q)) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (474)$$

in the q space, the locus of united points of functions of the three-system determined by points in the plane. All these quartics intersect in the critical curve (437).

In like manner to a line in the p space corresponds the twisted sextic curve

$$[q, f(p_1q), \ f(p_2q)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (475),$$

the locus of united points of a two-system.

The locus of Jacobian correspondents of points in the plane is the sextic curve

$$[f(p_1q), \ f(p_2q), \ f(p_3q)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (476).$$

Now any one of these sextics is the residual of the critical curve in the intersection of a pair of Jacobian quartics, and a curve meets its residual in t points, where

$$r + t = m(\mu + \nu - 2) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (477).$$

In particular for $r = 40$, $m = 10$, $\mu = \nu = 4$, we have $t = 20$; and so there are

twenty intersections, but I propose to show that these in reality correspond to ten contacts.

Take, for example, the curve (476), and let q_1 be a point of intersection and take p_1 to be the Jacobian correspondent of q_1, so that $f(p_1, q_1) = 0$. Then the tangent to the curve at q_1 is

$$[f(p_1 q),\ f(p_2 q_1),\ f(p_3 q_1)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (478).$$

But this tangent lies in the tangent planes at the same point to the system of quartics $(f(p_1 q), f(p_2 q), f(p_3 q), f(p_4 q) + uq) = 0$, where u is arbitrary, and as these quartics contain the critical curve, the sextics touch this curve where they meet it.

122. Hence, *the locus of the Jacobian correspondents of points on the critical curve is a curve of the tenth degree;* for in the plane $[p_1 p_2 p_3]$ there are ten points which are Jacobian correspondents of points on the critical curve.

The Jacobian quartic of the plane $[p_1 p_2 p_3]$ *contains ten lines.*

The tangent plane to the Jacobian quartic at a point on the critical curve, corresponding to one of the ten points just mentioned, intersects the plane of Art. 120 in a line which transforms into a plane cubic on the sextic surface into which the tangent plane to the quartic transforms. But the quartic transforms into a tangent plane to this sextic, and therefore contains the cubic, consequently the quartic contains the line.

123. We shall now consider the orders of the surfaces and curves into which given surfaces and curves are transformed by the relation connecting p and q (434).

With an arbitrary surface $Q = 0$ in the q space is associated a complementary $Q' = 0$, so that the points of the two surfaces compose tetrads of united points of functions of the four-system. These two surfaces, of orders m and m' respectively, transform into a common surface of order n.

An arbitrary line in the p space cuts the surface (n) in n points, and to these correspond $4n$ points in the q space situated on a sextic curve (475). This curve cuts the surface Q in $6m$ points, and these are generally united points of $6m$ distinct functions, because the surface Q is arbitrary. Hence $n = 6m$.

Again, the sextic cuts the surface Q' in $6m'$ points, but these fall into triads of united points complementary to the $6m$ points. Hence $n = \frac{1}{3}\, 6m'$; and we have the complete formula

$$n = 6m = 2m' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (479).$$

More generally, if the surface Q is wholly composed of sets of ν united points,

$$n = \frac{6m}{\nu} = \frac{6m'}{4 - \nu} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (480).$$

There is a case of exception for a Jacobian quartic $(q,\ f(p_1 q),\ f(p_2 q),\ f(p_3 q)) = 0$ which transforms into a plane and not a surface of the sixth degree as (480) would give for $\nu = m = 4$. But here the sextic curve cuts the quartic in 4 points and

touches it in 10 points on the critical curve (Art. 121), and the four points correspond to the intersection of the line with the plane in the p space, while to the ten points correspond lines of the type mentioned in Art. 109. We learn, therefore, that an arbitrary right line in the p space intersects ten of these lines, and that they compose a critical surface of the tenth order. This is otherwise justified from the consideration that an arbitrary quartic transformation converts a plane into a surface of the sixteenth order; and the fact that a plane transforms into a sextic shows that a critical surface of the tenth order has been discarded.

The equation of the complementary of the Jacobian $J(q) = 0$ will be found in Art. 127.

124. In like manner, taking an arbitrary curve in the q space of order M, let its complementary be of order M′, and let both transform into a curve of order N. The curve, being arbitrary, will not intersect the critical curve, and the 4M points in which it cuts the quartic, transformed from an arbitrary plane in the p space, will correspond point for point to the N points in which the transformed curve cuts the plane. Thus $N = 4M$.

Consider further the intersections of the curve and its complementary with an arbitrary surface (m) and its complementary (m'). The curve meets the complementary of the surface in Mm' points, and the complementary of the curve meets the surface in $M'm$ points. In general, each point of one set corresponds to one point of the other set, and the two sets compose pairs of united points. Thus $Mm' = M'm$, or $M' = 3M$ by (479); and accordingly we have the complete formula

$$N = 4M = \frac{4M'}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (481).$$

The whole set of points of intersection of the curve and surface and their complementaries is arranged as follows:—The Mm points unite with $3Mm$ of the $M'm'$ points in Mm tetrads. The Mm' points and the $M'm$ unite with $2(Mm' + M'm)$ of the $M'm'$ points to form tetrads, and thus by (481) and (479) all the $M'm'$ points are exhausted; and there are but $4Mm$ $(= Mm + Mm' + M'm)$ tetrads. But the curve (N) intersects the surface (n) in $Nn = 4M \times 6m$ points, and consequently there remain over 20 Mm points, which are critical points on the transformed curve and surface. These points evidently must lie on the critical surface of Art. 123.

When a curve is wholly composed of pairs of united points, the order of the transformed curve is $N = 2M$, and from symmetry the order of the complementary is $M' = M$.

An arbitrary surface and its complementary do not intersect in a curve wholly composed of pairs of united points, though of course the curve of intersection will contain all the pairs of united points which lie on the surface. It does not seem to be easy to assign any general relation connecting the order of a curve of this nature with that of its transformed curve. Thus 7 is the order of the curve transformed from the cubic intersection of a plane with its complementary (Art. 116).

125. We may account for the curve of intersection of the pair of sextics derived from two arbitrary planes in the following manner.

Call the two planes P and P′, and their complementary cubics C and C′. The complementary of the line (PP′) forms part of the intersection of the cubics C and C′, and this curve is a cubic (481). There remains, therefore, a sextic as part of the intersection of C and C′. The complementary of the cubic curve (PC′) is a curve of the ninth order, part being the cubic (P′C), and the remaining part the residual sextic on C and C′. This sextic is wholly composed of pairs of united points. The line and its complementary cubic transform into a common quartic. The cubic (PC′), the cubic (P′C) and the residual sextic transform into a common curve of order $3 \times 4 = 2 \times 6 = 12$ (compare the last article). Thus we can only account for a curve of order 16 $(= 4 + 12)$, and the sextics consequently intersect in a singular curve of order 20.

126. *The complex of lines joining pairs of united points is of the fourth order.*

If a and b are any two points on a line joining united points,

$$f(p, a) = xa + yb, \quad f(p, b) = za + wb \quad . \quad . \quad . \quad . \quad . \quad (482),$$

where p determines the function. The theory of quaternion arrays allows us to write the condition that these two equations should be simultaneously satisfied in the form*

$$\left\{\begin{matrix} f(e_1a) & f(e_2a) & f(e_3a) & f(e_4a) & a & b & 0 & 0 \\ f(e_1b) & f(e_2b) & f(e_3b) & f(e_4b) & 0 & 0 & a & b \end{matrix}\right\} = 0 \quad . \quad . \quad (483)$$

where e_1, e_2, e_3, e_4 are arbitrary quaternions; and by the rules of expansion of arrays, this equation is equivalent to

$$\Sigma \pm (f(e_1a), f(e_2a), a, b) \quad (f(e_3b), f(e_4b), a, b) = 0 \quad . \quad . \quad . \quad (484),$$

where the signs follow the rules of determinants. As this is of the fourth order in a and b, and also combinatorial with respect to both, it represents a complex of the fourth order.

127. By (433) and (434) we have

$$f(pq) = qJ(q), \quad p = F_q(q) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (485);$$

and throughout this article we shall suppose p expressed as a quartic function of q.

One root of the latent quartic of $f(pq)$ is thus equal to $J(q)$, so that when we substitute in the equation of that quartic (Art. 103 (420)), we have identically

$$J(q)^4 - J(q)^3 I'''(p) + J(q)^2 . I''(p) - J(q) I'(p) + I(p) = 0 \quad . \quad (486).$$

* The equations of the various assemblages of chords of Art. 113 may also be discussed by the aid of arrays.

The direct interpretation of this identity is that the transformation converts the Jacobian $I(p) = 0$ into two surfaces, one being the Jacobian $J(q) = 0$ and the other the surface of the twelfth order

$$J(q)^3 - J(q)^2 I'''(p) + J(q) I''(p) - I'(p) = 0 \quad . \quad . \quad . \quad (487).$$

This surface is the locus of three of the united points of functions which have a zero latent root, the fourth united point lying on the Jacobian $J(q) = 0$.

The critical curve is triple upon this surface, and the surface meets the Jacobian again in a residual curve of the eighteenth order, which is the locus of *united points corresponding to a double zero root.*

128. Making the substitution $s = t - J(q)$ in the latent quartic of the function $f(p, q)$ the equation reduces to

$$s^4 + s^3(4J(q) - I'''(p)) + s^2(6J(q)^2 - 3I'''(p) J(q) + I''(p))$$
$$+ s(4J(q)^3 - 3I'''(p) J(q)^2 + 2I''(p) J(q) - I'(p)) = 0 \quad . \quad (488).$$

A second root of the original quartic is equal to $J(q)$ if

$$4J(q)^3 - 3I'''(p) J(q)^2 + 2I''(p) J(q) - I'(p) = 0 \quad . \quad . \quad . \quad (489),$$

and this is the *locus of united points which correspond to double latent roots.* This surface is of the twelfth order, the critical curve is a triple curve upon it, and it meets the Jacobian in the same curves as (487).

The locus of united points corresponding to triple latent roots is the curve of intersection of this surface with the surface of the eighth order

$$6J(q)^2 - 3I'''(p) J(q) + I''(p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (490).$$

But the critical curve is double on this surface, and accordingly it counts six times in the intersection, so that the *locus of triple united points* is a curve of order $36 (= 8 \times 12 - 6 \times 10)$.

129. Further, quadruple united points are the points common to the surfaces (489), (490), and

$$4J(q) - I'''(p) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (491),$$

which do not lie upon the common critical curve.

In order to calculate the number of these quadruple points it is necessary to find the number of points common to the critical curve and the curve locus of triple points. Now $24 (= 4 \times 3 \times 2)$ functions have triple zero roots, this being the number of points common to the surfaces $I(p) = 0$, $I'(p) = 0$, $I''(p) = 0$ in the p space; and the curve locus of triple points being of the 36^{th} order meets $J(q) = 0$ in 144 points. Subtracting 24, there remain 120 points on the critical curve.

The triple curve therefore intersects (491) in 24 quadruple united points, and in

120 points on the critical curve; and thus *twenty-four functions of the system have four equal latent roots* and four coalesced united points.

130. Again, suppose that two roots of (488) are zero and that the remaining two are equal. In this case

$$8\mathrm{J}(q)^2 - 4\mathrm{J}(q)\,\mathrm{I}'''(p) + 4\mathrm{I}''(p) - \mathrm{I}'''(p)^2 = 0 \quad . \quad . \quad . \quad (492);$$

and this equation, combined with (489), gives a curve locus of order 36 ($= 8 \times 12 - 2 \times 3 \times 10$), which is the *locus of united points of functions whose roots are equal in pairs.*

We have now outlined the general theory of the four-system, but in a later section some supplementary remarks will be made on this subject.

SECTION XVIII.

The Quadratic Transformation of Points in Space.

The Second Example of the Use of the Bilinear Function.

131. The general quadratic transformation in space is represented by the equation

$$p = f(qq) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (493),$$

in which it is obviously permissible to regard the bilinear function as *permutable*, or the four linear functions (409) as self-conjugate. The transformation involves 40 constants.

To a plane in the p space corresponds a quadric, or

$$S l p = 0, \quad S l f(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (494)$$

transform one into the other; and thus to one point p correspond eight points q—the intersections of three quadrics—and to one point q corresponds in general one point p.

We use the word *octad* to denote the group of eight points corresponding to p.

132. The right line $q = a + tb$ transforms into the conic

$$p = f(aa) + 2tf(ab) + t^2 f(bb) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (495),$$

and $f(aa)$ and $f(bb)$ are two points on the conic, while $f(ab)$ is the pole of their chord.

The condition for the collinearity of these three points is

$$[f(aa), \quad f(ab), \quad f(bb)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (496);$$

and this equation may be replaced by

$$f(aa) + (x + y)f(ab) + xyf(bb) = 0, \quad \text{or} \quad f(a + xb, a + yb) = 0 \quad . \quad (497);$$

and this expresses that the original line joins Jacobian correspondents. *Thus lines joining Jacobian correspondents transform into lines.*

In this case (Art. 104) of the permutable function, if

$$f(rr') = 0 = f(r'r) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (498),$$

the points r and r' are conjugate to every quadric of the system (494).

We may replace (498) by

$$f(r \pm tr', r \pm tr') = f(rr) + t^2 f(r'r') \quad . \quad . \quad . \quad . \quad . \quad . \quad (499),$$

or *points harmonically conjugate to a pair of Jacobian correspondents transform into a single point.*

Thus we may speak of the rays of the assemblage of lines represented by (496) as *connectors,* (1) of a pair of Jacobian correspondents, (2) of a pair of points of an octad, (3) of an infinite number of pairs of points of octads.

It is evident that when two points of an octad coincide, they unite on the Jacobian; and that every point on the Jacobian is the union of a pair of points of an octad.

133. *The Jacobian correspondents transform into limiting points, separating the points derived from real from those derived from imaginary points.*

The points on the transformed connector

$$p = f(rr) + sf(r'r') \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (500)$$

are transformed from the points $r \pm \sqrt{s}\, r'$; these latter are real if s is positive; otherwise they are imaginary.

To discriminate between the *outer* and the *inner* region on the line (500), observe that the vectors from the centre of reciprocation to the limiting points are

$$\rho = \frac{\mathrm{V}f(rr)}{\mathrm{S}f(rr)}, \qquad \rho' = \frac{\mathrm{V}f(r'r')}{\mathrm{S}f(r'r')} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (501);$$

and that the vector to the point p is

$$\mathrm{OP} = \frac{\mathrm{V}f(rr) + s\mathrm{V}f(r'r')}{\mathrm{S}f(rr) + s\mathrm{S}f(r'r')} = \frac{\rho\mathrm{S}f(rr) + s\rho'\mathrm{S}f(r'r')}{\mathrm{S}f(rr) + s\mathrm{S}f(r'r')} \quad . \quad . \quad . \quad (502).$$

The point P lies on the inner region if $\mathrm{S}f(rr)$ and $s\mathrm{S}f(r'r')$ are of like sign; and the inner region corresponds to real points if the points r and r' are either both inside or both outside the quadric

$$\mathrm{S}f(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (503).$$

This quadric is the locus of points projected to infinity; it may of course be imaginary, so that $\mathrm{S}f(rr)$ and $\mathrm{S}f(r'r')$ are essentially one-signed if r and r' are real. In this case the region is always inner. If the quadric is real, the points r and r' (if real) cannot both lie inside, for they are conjugate to it. The nature of the intersection of a line with this quadric controls the nature of the conic into which it is transformed.

134. The locus of the Jacobian correspondents of points in a plane is a sextic curve, and for the permutable function this sextic cuts an arbitrary plane in points which correspond in pairs. There are therefore three connectors in a plane.

The vertices of the triangle of connectors belong to the same octad; for if q_1 is one vertex and q_2 and q_3 the points, one on each of the connectors through q_1, which (Art. 132) belong to the same octad as q_1, then q_2 and q_3 belong to a common octad, and their line is a connector—the third connector in the plane.

We may suppose the weights of the points q_1, q_2 and q_3 chosen so that the Jacobian correspondents are

$$q_2 \pm q_3, \quad q_3 \pm q_1, \quad q_1 \pm q_2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (504),$$

the vertices of the triangle being (Art. 132) harmonically conjugate to these points in pairs.

135. Let the eight quaternions which represent points of an octad have their weights chosen so that*

$$p_0 = f(q_1 q_1) = f(q_2 q_2) = \&\text{c.} = f(q_8 q_8) \quad . \quad . \quad . \quad . \quad . \quad (505),$$

* It follows from Art. 132, that this convention is the same as that made at the end of the last article.

and let the twenty-eight points $f(q_1 q_2)$ be denoted by

$$p_{12} = f(q_1 q_2), \text{ \&c. } \quad p_{78} = f(q_7 q_8) \quad . \quad . \quad . \quad . \quad . \quad (506).$$

It may be remarked that these relations lead to

$$\pm 2\sqrt{-1} p_{12} = f(q_1 \pm \sqrt{-1} q_2, \quad q_1 \pm \sqrt{-1} q_2) \quad . \quad . \quad . \quad (507);$$

so that the points (506), although real, if the points of the octad are real, have been transformed from imaginary points, and consequently do not lie in the same region (Art. 133) as the point p_0.

The Jacobian correspondents transform into $p_0 \pm p_{12}$, &c.

136. *A plane transforms into a* STEINER'S *quartic.*

In the notation of the last article, the plane

$$q = t_1 q_1 + t_2 q_2 + t_3 q_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (508)$$

transforms into the surface

$$p = p_0 (t_1^2 + t_2^2 + t_3^2) + 2p_{23} t_2 t_3 + 2p_{31} t_3 t_1 + 2p_{12} t_1 t_2 \quad . \quad . \quad . \quad (509);$$

and if we write the identity connecting the five quaternions in the form

$$p = p_0 w + p_{23} x + p_{31} y + p_{12} z \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (510),$$

comparison with (509) gives

$$2xyzw = y^2 z^2 + z^2 x^2 + x^2 y^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (511)$$

on elimination of the parameters t. This is the scalar equation of the surface (509), and the existence of the three intersecting double lines (y, z; z, x; and x, y), which characterize a STEINER'S quartic, is manifest.

Evidently the three connectors transform into the double lines; and the points $p_0 \pm p_{23}$, $p_0 \pm p_{31}$, $p_0 \pm p_{12}$ separate (Art. 133) the lines into regions intersected by a pair of real and a pair of imaginary sheets of the surface.*

137. The nature of the surface into which a plane transforms may be established from purely geometrical considerations. A tangent plane to the surface transforms back into a quadric touching the plane, that is, cutting it in a pair of lines. These lines transform back into conics in the tangent plane and on the surface. One point of intersection of these conics corresponds to the point of intersection of the lines. The other three points must result from the union of pairs of points of octads, and therefore the lines must cut the sides of the triangle in points harmonically conjugate to the Jacobian correspondents. The conics consequently intersect the lines into which the three connectors transform, and these three lines must be double. In terms

* It is easy to verify this by determining the greatest and least value of $t_2 t_3 (t_2^2 + t_3^2)^{-1}$ for real values of t_2 and t_3. Compare (509).

of the parameters, the equations of a pair of lines transforming into conics in a common plane must be

$$u_1t_1 + u_2t_2 + u_3t_3 = 0, \quad \frac{t_1}{u_1} + \frac{t_2}{u_2} + \frac{t_3}{u_3} = 0 \quad . \quad . \quad . \quad . \quad . \quad (512);$$

this is a consequence of the harmonic section. Two lines thus related may be said to be conjugate, and there exist four self-conjugate lines

$$t_1 \pm t_2 \pm t_3 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (513),$$

any one of which transforms into a conic having ring-contact with the quartic. The planes of these four conics transform back into cones, touching the plane along the self-conjugate lines. The self-conjugate lines join triads of non-corresponding Jacobian points, such as $q_1 + q_2$, $q_2 + q_3$, $q_3 - q_1$.

It is easy to see that the four conics are inscribed to the faces of a tetrahedron, and that each touches the other three. Consider, for example, the conics transformed from the sides of the triangle, $q_2 + q_3$, $q_3 + q_1$, $q_1 + q_2$. The equation of one conic is

$$\begin{aligned} p &= f(q_2 + q_3 + t(q_3 + q_1), \quad q_2 + q_3 + t(q_3 + q_1)) \\ &= 2(p_0 + p_{23}) + 2t(p_0 + p_{23} + p_{31} + p_{12}) + 2t^2(p_0 + p_{31}) \quad . \quad . \quad (514); \end{aligned}$$

and this shows that the conic passes through a limiting point on each of two of the double lines; and as the pole of the chord is symmetrical with respect to the suffixes, it is likewise the pole of corresponding chords for the conics into which the other sides of the triangle transform.

It is not difficult to prove that every line in the plane through one of the six Jacobian points transforms into a conic having a fixed tangent. The tangent for the point $q_1 + q_2$ is

$$p = p_0 + p_{12} + t(p_{23} + p_{31}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (515).$$

138. Let a connector meet the Jacobian in the points a, a', b and c, a and a' being correspondents so that $f(aa') = 0$; let b' and c' be the correspondents of b and c; and consider the points of an octad in the plane $[b'aa']$. The two connectors aa' and bb' in this plane intersect in the point b, and as b is its own harmonic conjugate with respect to b and b', two sides of the triangle of Art. 134 unite in the line aa'. Let $b_{,}$ be the harmonic conjugate of b with respect to a and a', then $b_{,}$ is a vertex of the infinitely slender triangle, the remaining two being the point b counted twice. (Compare Arts. 132 and 134.)

The point $b_{,}$, being the intersection of the connector aa' with a consecutive connector, is a *focal point* on the ray aa' of the congruency (496); and similarly $c_{,}$, the harmonic conjugate of c to a and a', is the second focal point; and by HAMILTON's theory the ray touches the focal surface at these two points.*

* This theorem of the construction of the focal points is an extension of Mr. RUSSELL's theorem for the congruency of lines joining corresponding points on the Hessian of a cubic surface. R. RUSSELL, "Geometry of Surfaces derived from Cubics," 'Proc. Roy. Irish Acad.,' vol. 5, p. 464.

In this case the plane transforms into the surface

$$p = f(t_1 b + t_2 b_{\prime} + t_3 b', \ t_1 b + t_2 b_{\prime} + t_3 b')$$
$$= t_1^2 f(bb) + t_2^2 f(b_{\prime} b_{\prime}) + t_3^2 f(b'b') + 2t_2 t_3 f(b_{\prime} b') + 2t_1 t_2 f(bb_{\prime}) \quad . \quad (516),$$

and if we take (as we may) $f(bb) = f(b_{\prime} b_{\prime})$, the scalar equation of the surface takes the form

$$4xy^2 w = 4x^2 z^2 + y^4, \text{ where } w = t_1^2 + t_2^2, \ x = t_3^2, \ y = 2t_2 t_3, \ z = 2t_1 t_2 \quad (517).$$

On comparison with (511) we see that two of the lines of the STEINER'S quartic have united; for $x = 0$ we have the line x, y counted four times.

139. By a process similar to that of Arts. 123 and 124, but much simpler, we can determine the order (m') of the complementary of a surface of order m, and the order (n') of the surface into which both transform. The formula is

$$\frac{4m}{\nu} = \frac{4m'}{8-\nu} = n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (518),$$

where ν is the number of points of octads of which the surface is wholly composed.* And this formula is proved without trouble, remembering that a line in the p space transforms into a twisted quartic—the intersection of two quadric surfaces.

In like manner† for a curve (M), its complementary (M′) and its transformed (N),

$$\frac{2\mathrm{M}}{\nu} = \frac{2\mathrm{M}'}{8-\nu} = \mathrm{N} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (519).$$

Thus the complementary of a connector is a twisted cubic;‡ the complementary of a plane is a surface of the seventh order, which cuts the plane in the triangle of connectors and in a quartic—probably the four lines of Art. 137.

The formulæ of this article are not directly applicable to the Jacobian, which is a critical surface of the transformation. The twisted quartic into which a line in the p space transforms, cuts the Jacobian in 16 points and does not in general touch it. For if it did the twisted quartic would have a double point. Consequently, the Jacobian transforms into a surface of the sixteenth order. Every point on the

* For the general transformation of order μ, the relation is

$$\frac{\mu^2 m}{\nu} = \frac{\mu^2 m'}{\mu^3 - \nu} = n.$$

† For a transformation of order μ,

$$\frac{\mu \mathrm{M}}{\nu} = \frac{\mu \mathrm{M}'}{\mu^3 - \nu} = \mathrm{N}.$$

‡ For example,

$$q = \sum_1^4 \frac{q_n}{x_n - ty_n}, \text{ where } q_5 = \sum_1^4 \frac{q_n}{x_n}, \ q_6 = \sum_1^4 \frac{q_n}{y_n},$$

is the equation of the twisted cubic through six points $q_1, q_2 \ldots q_6$, and it is not difficult to verify that this curve and the line $q = q_7 + tq_8$ transform into a common line $p = p_0 + tp_{78}$ if the eight points form an octad.

Jacobian is the union of a pair of points of an octad (Art. 132), and therefore the complementary surface is composed of hexads of points of octads, and its order is consequently 24, or six times that of the Jacobian, because the quartic cuts it in a hexad for every point of intersection with the Jacobian.

140. *The complementary of the Jacobian is the focal surface of the congruency of connectors.**

When two points of a set transforming into a common point approach coincidence, they close in on the Jacobian, and simultaneously the remaining points of the set reach the complementary surface. Through any one of these remaining points two consecutive connectors pass; and therefore, by HAMILTON's beautiful theory, the remaining points are *focal points* on the rays connecting them to the coincident points.

Every ray touches the focal surface in two points—the two focal points on the ray; and for a quadratic transformation it cuts that surface in twenty other points. *These twenty points are harmonically conjugate in pairs to the Jacobian correspondents.* For (Art. 132) the harmonic conjugate of any one of the points belongs to the same octad as that point; but the focal surface is complementary and is wholly composed of hexads of points of octads, and therefore the harmonic conjugate is also on the focal surface.

141. *The focal surface of the transformed connectors is the transformed Jacobian.*

On transformation the harmonic conjugates on a connector unite. In the notation of Art. 138, the point b and the focal point $b_{\prime}$ unite in a focal point of the transformed connector, for through $b_{\prime}$ pass two consecutive connectors which transform into consecutive connectors through $f(b_{\prime}b_{\prime})$. Similarly the points c and $c_{\prime}$ transform into the second focal point and the transformed Jacobian is consequently the focal surface. The twenty points of the last article transform into ten points. The Jacobian correspondents a and a' transform into limiting points (Art. 133). Thus we have accounted for the sixteen points in which the transformed connector meets its focal surface.

The class of the transformed Jacobian is $n' = 4$. In the p space draw a plane through an arbitrary line to touch the surface. This plane contains a pair of consecutive transformed connectors, and on passing back to the q space it becomes a quadric containing consecutive intersecting connectors. This quadric is therefore a cone. The system of planes through the arbitrary line transforms into a system of quadrics through a twisted quartic, and four of these quadrics are cones. To these four cones correspond four tangent planes to the focal surface through the arbitrary line. Hence we may write down the equation of the reciprocal of the transformed Jacobian. The condition that the quadric $Slf(qq) = 0$ should be a cone† is

* This theorem is true for the connectors of a set of points to a coincident pair of the set for all transformations.

† If $f(qq) = \Sigma a_1 Sqf_1q$, then $f'(lq) = \Sigma f_1 q Sla_1$.

$$f'(lq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (520),$$

where r is the vertex, for the tangent plane $\mathrm{S}lf(qr) = \mathrm{S}qf'(lr) = 0$ must vanish identically. Hence the fourth invariant of $f'(lr)$ must vanish, or

$$(f'(la),\ f'(lb),\ f'(lc),\ f'(ld)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (521),$$

and this is the equation of the reciprocal of the surface.

Thus the transformed Jacobian is the reciprocal of a Jacobian surface, but one of less generality than those previously considered. We may replace (520) by four equations

$$\mathrm{S}lf(ra) = 0, \quad \mathrm{S}lf(rb) = 0, \quad \mathrm{S}lf(rc) = 0, \quad \mathrm{S}lf(rd) = 0 \quad . \quad . \quad . \quad (522);$$

and because f is a permutable function, on replacing r by $xa + yb + zc + wd$ and eliminating x, y, z and w, we obtain the symmetrical determinant

$$\begin{vmatrix} \mathrm{S}lf(aa), & \mathrm{S}lf(ab), & \mathrm{S}lf(ac), & \mathrm{S}lf(ad) \\ \mathrm{S}lf(ab), & \mathrm{S}lf(bb), & \mathrm{S}lf(bc), & \mathrm{S}lf(bd) \\ \mathrm{S}lf(ac), & \mathrm{S}lf(bc), & \mathrm{S}lf(cc), & \mathrm{S}lf(cd) \\ \mathrm{S}lf(ad), & \mathrm{S}lf(bd), & \mathrm{S}lf(cd), & \mathrm{S}lf(dd) \end{vmatrix} = 0 \quad . \quad . \quad . \quad (523).$$

But ('Three Dimensions,' Art. 528) a surface, whose equation is a symmetrical determinant with constituents linear in the variables, has ten double points. This accounts for the class of the surface being 16 instead of 36 ($= 4(4-1)^2$).

In the case in which the function is self-conjugate as well as permutable, that is when p, q and r may be transposed in $\mathrm{S}pf(qr)$ in any manner, we have the theory of the corresponding points on the Hessian of the general cubic surface

$$\mathrm{S}qf(qq) = 0$$

and Mr. RUSSELL's paper may be referred to for various examples.

142. The characteristics of the two congruencies are found thus. The order of the congruency of connectors is obviously $\mu = 7$, as seven connectors can be drawn from an arbitrary point to the remaining points of the octad to which the point belongs. The class is $\nu = 3$, for three connectors lie in a plane. The order of the focal surface (Art. 139) is $\mathrm{M} = 24$. Its class is $\mathrm{N} = 16$. This follows from the relation ('Three Dimensions,' Art. 510)

$$\mathrm{M} - \mathrm{N} = 2(\mu - \nu) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (524);$$

or independently by Mr. RUSSELL's elegant method* which is applicable in this more general case.

For the transformed congruency, the order is $\mu' = 28$ (Art. 135), the order of the focal surface is $\mathrm{M}' = 16$, and its class is $\mathrm{N}' = 4$ (Arts. 139, 141); and therefore (524) the class of the congruency is $\nu' = 22$.

* 'Proc. Roy. Irish Acad.,' vol. 5, p. 473.

Consequently twenty-two connectors are generators of a quadric $Slf(qq) = 0$; and in particular the polar quadric of a point with respect to a cubic surface contains 22 generators joining corresponding points on the Hessian.

SECTION XIX.

Homography of Points in Space.

The Third Example of the Use of the Bilinear Function.

143. Writing generally

$$f(pq) = r \quad \text{or} \quad \{f(pq), r\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (525),$$

and regarding r as a constant quaternion, a one-to-one relation is established between the points p and q, so that one may be said to be the homograph of the other.

This is equivalent to three relations of the form

$$Spf_1q = 0, \quad Spf_2q = 0, \quad Spf_3q = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (526);$$

and accordingly the bilinear function is not utilized to its full extent, but it seems to be the most convenient instrument for investigating the subject.

144. We have generally in the notation of Arts. 107, 108,

$$qI(p) = F_p(r), \quad pJ(q) = F_q(r) \quad . \quad . \quad . \quad . \quad . \quad . \quad (527),$$

and thus the critical curves of the transformation are

$$F_p(r) = 0 \quad \text{and} \quad F_q(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (528)$$

respectively; or (compare (437))

$$((r, f(pa), f(pb), f(pc), f(pd))) = 0 \text{ and } ((r, f(aq), f(bq), f(cq), f(dq))) = 0 \quad . \quad (529).$$

These curves are sextics, and because (528) may be replaced by

$$[f''(pr_1), f''(pr_2), f''(pr_3)] = 0, \quad [f'(r_1q), f'(r_2q), f'(r_3q)] = 0 \quad . \quad (530),$$

where $[r_1r_2r_3] = 0$, they may be described as the locus of Jacobian correspondents of points in the plane reciprocal to the point r (424).

As in Art. 109, when a point (q) is on the critical curve, its homograph is a line

$$pJ'(q) = tG_q(r) + F_q(p), \quad F_q(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad (531),$$

and not a point; and as in Art. 112 the homograph of a line $q + xq'$ is a twisted cubic

$$p = (p_0p_1p_2p_3 \between x1)^3 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (532);$$

and a line of the type (531) breaks off the cubic for every intersection with the critical curve.

Thus, when the line is a chord of the critical curve, its homograph is also a line, so that

$$\{f(p + xp', \ q + xq'), r\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (533).$$

Symmetry shows that $p + xp'$ must be a chord of the second critical curve.

If the homograph of a line is plane, it is at most a conic. For the condition of planarity (compare Art. 120)

$$(p_0p_1p_2p_3) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (534)$$

is of the sixth order in q and in q', and this equation represents a complex of the sixth order. But this complex can include nothing except intersectors of the critical sextic, for the cone of intersectors from the arbitrary point q is of the sixth order.

The ruled surface of triple chords has been noticed in Art. 75.

145. The homograph of a plane

$$Slq = 0 \quad \text{is} \quad SlF_p(r) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (535),$$

a general cubic surface through the critical curve.

This cubic surface also passes through the sextic

$$F_p'(l) = 0 . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (536),$$

and it intersects the Jacobian $I(p) = 0$ in this sextic and in the critical curve. The equation of the Jacobian may be written in the forms

$$Sf_p'(l)\, F_p(r) = Sf_p(r)\, F_p'(l) = I(p)\, Slr = 0 . \quad . \quad . \quad . \quad . \quad (537),$$

and for l and r, both variable, the curves $F_p(r) = 0$, $F_p'(l) = 0$ generate the Jacobian in a manner analogous to the double generation of a quadric. Since the rank of the sextic is $r = 16$ (Art. 64), the two curves intersect in 14 points (477).

146. It may be of interest to show how we can fully account for the lines on the cubic surface (535). Let the six points in which the critical curve $F_q(r) = 0$ cuts the plane $Slq = 0$ be denoted by the symbols 1, 2, 3, 4, 5, 6; and let (12), (23), &c., denote the fifteen connectors of these points. Further let [1], [2], . . . [6] denote the six conics that can be drawn through all but one of the six points.

The curves and points represented by these 27 *symbols transform into the lines on the cubic.* By (531) and (533) we account for the lines and the points. In general a unicursal curve transforms into a curve of thrice the order, but for every intersection with the critical curve a line breaks off. Thus the six conics likewise transform into lines.

Any pair of these loci, which intersect in a point which is not critical, continue to intersect after transformation, and this consideration enables us to write down the full scheme of double-sixes on the cubic surface. These fall into three types:—

$$\text{I.}\quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ [1] & [2] & [3] & [4] & [5] & [6] \end{pmatrix}.$$

$$\text{II.}\quad \begin{pmatrix} 1 & 2 & 3 & (56) & (64) & (45) \\ (23) & (31) & (12) & [4] & [5] & [6] \end{pmatrix}.$$

$$\text{III.}\quad \begin{pmatrix} 1 & [1] & (23) & (24) & (25) & (26) \\ 2 & [2] & (13) & (14) & (15) & (16) \end{pmatrix}.$$

In these schemes, every line represented by a symbol in one row intersects every line in the other row, except that denoted by the symbol in the same column. There are thus 36 double-sixes; one of the first type, twenty of the second, fifteen of the third.

The schemes are easily obtained by taking two non-intersecting lines, say 1 and [1], when we have

$$\begin{array}{lll} 1 & \text{intersects} & (12), (13), (14), (15), (16), [2], [3], [4], [5], [6], \\ [1] & \quad ,, & (12), (13), (14), (15), (16), \ 2, \ 3, \ 4, \ 5, \ 6, \end{array}$$

and, discarding the common lines, the double-six is found. In like manner the 45 triple tangent planes belong to one or other of the types

$$(1, [2], (12)) \quad \text{or} \quad ((12), (34), (56)).$$

147. One or two relations respecting a point on a critical curve and its line homograph may be mentioned. Since the line (531) has a point for its homograph, it must be a triple chord of the sextic $F_p(r) = 0$. It meets this sextic in three points, p_1, p_2, p_3, and intersects the Jacobian in a fourth point p_0 or $F_q(p)$. To the three points p_1, p_2, p_3 correspond the three triple chords of the q sextic which pass through q; and the homograph of every plane through the line p_1, p_2, p_3 is a cubic

having q as a double point and containing the three triple chords which pass through q.

The cubic homograph of any plane contains the critical sextic which counts thrice in its intersection with the octic surface of triple chords, and the remainder of the intersection consists of the six line-homographs of the critical points in the plane.

The homograph of the surface of chords of the p sextic, which meet the line $p_1p_2p_3$, is the cone whose vertex is q and which contains the q sextic.

The homograph of one sextic is the surface of triple chords of the other.

One chord can be drawn to meet two non-intersecting triple chords in points not on the sextic. Its homograph is the line joining the homographs of these chords.

The locus of the points $F_q(p)$, the Jacobian correspondents of points on the critical curve, is a curve of the fourteenth order. For the octic surface intersects the Jacobian in the second critical curve counted thrice, and in a residual curve of order 14.

148. Connectors of points with their homographs compose the complex of the sixth order

$$(f(pp), f(pq), f(qp), r)\,(f(pq), f(qp), f(qq), r)$$
$$= (f(pp), f(pq), f(qq), r)\,(f(pp), f(qp), f(qq), r) \quad . \quad (538),$$

as appears on elimination of x, y, z and w from

$$f(xp + yq,\ zp + wq) = r \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (539).$$

Or in other words, this is the assemblage of lines which meet their twisted cubic homographs.

The condition that two pairs of homographs should be on the same line is

$$((f(pp), f(pq), f(qp), f(qq), r)) = 0 \quad . \quad . \quad . \quad . \quad . \quad (540),$$

for if two sets of values of x, y, z, w satisfy (539), the five quaternions included in (540) must be co-planar. Now (540) imposes two conditions on the line pq, and therefore represents a congruency of lines; and from the conditions implied in (540) we can select but two combinatorial functions with respect to p and q. These are

$$(f(pp), f(pq), f(qp), f(qq)) = 0,\ (f(pp), f(pq) + f(qp), f(qq), r) = 0 \quad . \quad (541);$$

and the congruency is therefore common to two complexes of the fourth and third orders respectively. But these complexes contain the congruency

$$[f(pp), f(pq) + f(qp), f(qq)] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (542),$$

and this is foreign to the question, being, in fact, the congruency (496) of Art. 132 of connectors for the permutable function $f(pq) + f(qp)$. When this is rejected, there remains the congruency of connectors of two pairs of homographs, and its order and

class are $\mu = 5\,(= 4 \times 3 - 7)$, $\nu = 9\,(= 4 \times 3 - 3)$, for the congruency (542) has been shown to be of the seventh order and third class.

Equations (541) being supposed satisfied, they are equivalent to

$$u_1 f(pp) + u_2 f(pq) + u'_2 f(qp) + u_3 f(qq) = 0,$$

$$v_1 f(pp) + v_2\,(f(pq) + f(qp)) + v_3 f(qq) = r \quad . \quad . \quad . \quad . \quad (543);$$

and multiplying the first by t and adding it to the second, we find that t must satisfy the quadratic

$$(v_2 + tu_2)(v_2 + tu'_2) = (v_1 + tu_1)(v_3 + tu_3) \quad . \quad . \quad . \quad . \quad (544),$$

if the sum can be reduced to the form (539). The roots of this equation lead to the determination of the two pairs of homographs.

The bi-connectors of homographs which pass through a point are double edges of the cone of connectors of homographs, and those which lie in a plane are bi-tangents to the curve enveloped by the connectors. This appears from the forms of the equations (538) and (540).

149. The congruency of connectors of Jacobian correspondents is intimately connected with the theory of the last article.

We have already considered the case in which the function is permutable, but matters now are much more complicated.

The congruency may be expressed by

$$f(pp) + uf(pq) + vf(qp) + uvf(qq) = 0 \quad . \quad . \quad . \quad . \quad . \quad (545),$$

and it is obvious that it is included in the quartic complex, the first of (541), and it is easy to verify that it is also included in the sextic complex (538) and that *no matter what quaternion "r" may be.* Replacing uv by w in (545) and substituting in the equations of these two complexes we find that either $w = uv$, or else the lines must belong to the congruency (540). In other words, the congruency of this article is complementary to the congruency of the last as regards the two complexes. But the rays of the former congruency count double as edges of cones or as tangents in planes. Hence the order and class of the congruency under discussion are $\mu = 14\,(= 4 \times 6 - 2 \times 5)$, $\nu = 6\,(= 4 \times 6 - 2 \times 9)$.

These numbers are exactly double the corresponding numbers for the permutable function, and as regards the class there is no difficulty in seeing how this arises. In general there are two sextic loci of Jacobian correspondents of the points in a plane (528), and the connectors in the plane join the six points of one to the corresponding six points of the other. For the permutable function the two loci coalesce, and the number of connectors is halved.

Again, we may say that the lines of this new congruency through a point are *fixed* edges of the cone (538), and the lines in a plane fixed tangents to a sextic curve,

because they are independent of r; the lines of the former congruency are double edges and double tangents.

We proceed to determine the class of the focal surface. The equations

$$(pqab) = 0, \quad f(pq) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (546)$$

require a ray to intersect the fixed line a, b. Eliminating p, the equation of the locus of q is

$$f(qq) + xf(aq) + yf(bq) = 0, \quad \text{or} \quad [f(qq), f(aq), f(bq)] \quad . \quad . \quad (547);$$

and this (274) is a curve of order $m = 11$ and rank $r = 48$. But this curve is a complex curve consisting of the line ab and a residual which intersects it in four points on the Jacobian. The order and rank of the residual are $m = 10$, $r = 40$, the rank being diminished by twice the number of intersections. The number (r) of tangent planes through ab to this curve minus twice the number of intersections gives the number of planes through ab containing consecutive rays. Thus the class of the focal surface is $\mathrm{N} = 32$, and its order (524) is $\mathrm{M} = 48$. Every one of these numbers is double the corresponding number obtained in Art. 142 for the permutable function.

For the sake of completeness we wish to show the nature of the assemblage of lines common to the complex (538) and the second complex (541), as we have already completely considered the lines common to the remaining two pairs. Evidently the congruency of bi-connectors belongs to these two complexes and is counted twice among their common lines. There remains an assemblage of lines of order $\mu = 3 \times 6 - 2 \times 5 = 8$, and of class $\nu = 3 \times 6 - 2 \times 9 = 0$. It is easy to prove by the method of this article that these lines join an arbitrary point to the eight correspondents of r in the quadratic transformation $f(pp) = r$.

SECTION XX.

The Method of Arrays.

Applications to n-Systems of Linear Functions.

150. We shall illustrate the method of quaternion arrays* by a few examples on systems of linear functions. These functions may be supposed to be of the most general kind, functions of a point in space of μ dimensions, but we pay particular attention to the case of three dimensions.

An array of n rows and m columns vanishes if, and only if, the constituents in the rows are connected by the same set of scalar coefficients $x_1, x_2 \ldots x_m$. Thus

$$\left\{\begin{matrix} a_1 & a_2 & a_3 & \ldots & a_m \\ b_1 & b_2 & b_3 & \ldots & b_m \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ l_1 & l_2 & l_3 & \ldots & l_m \\ p_1 & p_2 & p_3 & \ldots & p_m \\ \cdot & \cdot & \cdot & & \cdot \\ \cdot & \cdot & \cdot & & \cdot \\ r_1 & r_2 & r_3 & \ldots & r_m \end{matrix}\right\} = 0 \quad \ldots \ldots \quad (548),$$

when

$$\Sigma x_s a_s = 0,\ \Sigma x_s b_s = 0,\ \ldots\ \Sigma x_s r_s = 0 \quad \ldots \ldots \quad (549).$$

It is proved in the memoir that the expansion of the array is of the form†

$$\Sigma \pm (a_1 a_2 a_3 a_4)(b_5 b_6 b_7 b_8) \ldots (l_{4n'-3},\ l_{4n'-2},\ l_{4n'-1},\ l_{4n'})$$
$$\times \left\{\begin{matrix} p_{4n'+1} & p_{4n'+2} & \ldots & p_m \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ r_{4n'+1} & r_{4n'+2} & \ldots & r_m \end{matrix}\right\} \quad \ldots \ldots \quad (550);$$

and we take definitely $m = 4n' + n''$, where $n'' = 0$, 1, 2 or 3. The number of equivalent scalar conditions is $4m - n + 1$ for the vanishing of a quaternion array, and $(\mu + 1)m - n + 1$ for an array of points in μ dimensions.

The scalars x_1, x_2, &c., are determined when (548) is satisfied by the system of arrays of $m - 1$ columns and n rows, of which this array

$$\left\{\begin{matrix} x_1 a_1 + x_2 a_2, & a_3, & \ldots & a_m \\ x_1 b_1 + x_2 b_2, & b_3 & & b_m \\ \cdot & & & \\ \cdot & & & \\ x_1 r_1 + x_2 r_2, & r_3 & \ldots & r_m \end{matrix}\right\} = 0 \quad \ldots \ldots \quad (551)$$

is a type.

* 'Trans. Roy. Irish Acad.,' vol. 32, pp. 17–30.

† Every row must be represented in the expansion, and it may be gathered from the Memoir how to expand if one row involves only four constituents. In this case the general method fails.

If all the minor arrays formed by omitting one column of (548) vanish, we take any two of these minors, and forming second minors corresponding to (551) we obtain two sets of relations (549), and so on in general.

151. In order to find the conditions that a linear function of an n-system should convert m given *weighted* points $a_1, \ldots a_m$ into m others, $b_1, \ldots b_m$, we write down the array in m rows and $n+1$ columns,

$$\left\{\begin{array}{cccccc} f_1a_1 & f_2a_1 & . & . & f_na_1 & b_1 \\ f_1a_2 & f_2a_2 & . & . & f_na_2 & b_2 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ f_1a_m & f_2a_m & . & . & f_na_m & b_m \end{array}\right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (552),$$

whose vanishing requires

$$\Sigma x_s f_s a_t = b_t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (553).$$

The vanishing of this array requires $4m - n$ scalar equations to be satisfied. If then $n = 4m$, the array vanishes without restriction, and a single condition must be satisfied for the vanishing of the arrays, such as (551),

$$\left\{\begin{array}{cccc} x_1f_1a_1 + b_1 & f_2a_1 & . \ . & f_na_1 \\ x_1f_1a_2 + b_2 & f_2a_2 & . \ . & f_na_2 \\ . & . & & . \\ . & . & & . \\ x_1f_1a_m + b_m & f_2a_m & . \ . & f_na_m \end{array}\right\} = 0, \text{ \&c.} \quad . \quad . \quad . \quad . \quad (554),$$

and these determine the coefficients x without ambiguity.

Thus from a given $4m$-system can be found one function which shall convert m given weighted points into other given weighted points. (Compare Art. 3.)

152. When the weights are disregarded, the equations of condition are

$$\Sigma x_s f_s a_1 = y_1 b_1, \quad \Sigma x_s f_s a_2 = y_2 b_2, \quad \ldots \quad \Sigma x_s f_s a_m = y_m b_m \quad . \quad . \quad . \quad (555);$$

and these furnish the array

$$\left\{\begin{array}{cccccccccc} f_1a_1 & f_2a_1 & . \ . \ . & f_na_1 & b_1 & 0 & 0 & . & . & 0 \\ f_1a_2 & f_2a_2 & . \ . \ . & f_na_2 & 0 & b_2 & 0 & . & . & 0 \\ . & . & & . & . & . & . & & & . \\ . & . & & . & . & . & . & & & . \\ f_1a_m & f_2a_m & . \ . \ . & f_na_m & 0 & 0 & 0 & . & . & b_m \end{array}\right\} = 0 \quad . \quad . \quad . \quad . \quad (556),$$

of $m + n$ columns and m rows. Its vanishing requires $3m - n + 1$ conditions to be satisfied, and the vanishing of the minor arrays such as (551) requires a single condition if $n = 3m + 1$, and these definitely determine the function. Thus from a $(3m + 1)$-system can be found one function which converts m points to m others when the weights are neglected. In particular, a linear transformation can be found (out of the whole sixteen-system) to convert five points into five others (Art. 3).

153. When lines are to be converted into lines, the conditions are

$$\Sigma x_s f_s a_t = y_t b_t + y'_t b'_t\,; \quad \Sigma x_s f_s a'_t = z_t b_t + z'_t b'_t \quad . \quad . \quad . \quad . \quad (557),$$

and the array

$$\left\{\begin{array}{cccccccccccc} f_1 a_1 & f_2 a_1 & \ldots & f_n a_1 & b_1 & b'_1 & 0 & 0 & \ldots & 0 & 0 & 0 \quad 0 \\ f_1 a'_1 & f_2 a'_1 & \ldots & f_n a'_1 & 0 & 0 & b_1 & b'_1 & \ldots & 0 & 0 & 0 \quad 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \quad \cdot \\ f_1 a_m & f_2 a_m & \ldots & f_n a_m & 0 & 0 & 0 & 0 & \ldots & b_m & b'_m & 0 \quad 0 \\ f_1 a'_m & f_2 a'_m & \ldots & f_n a'_m & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & b_m \quad b'_m \end{array}\right\} \quad . \quad . \quad (558)$$

of $n + 2m$ columns and of $2m$ rows must vanish. The number of conditions is $6m - n + 1$. Thus a function of a seven-system and of a thirteen-system respectively converts one and two lines into one and two others.

In like manner, when planes are to be converted into planes, the array is of $n + 3m$ columns and of $3m$ rows, and requires $9m - n + 1$ conditions for its vanishing.

In general for space of μ dimensions a function of an n-system is completely defined if

$$n = \mu\,(m_1 + 2m_2 + 3m_3 + \&c.) + 1 = \mu \mathrm{M} + 1 \quad . \quad . \quad . \quad . \quad (559),$$

which converts m_1 given points, m_2 lines, m_3 planes, &c., into other given points, lines and planes, &c.

154. We shall now suppose that the array (556) does not vanish without conditions restricting the generality of the points. Let all the points except a_m and b_m be given. It is sufficient to consider the cases in which the number of conditions does not exceed three.

By the expansion (550) we have, if $3m - n + 1 = \nu$, so that ν conditions must be satisfied, or if $n = 3m - \nu + 1 = 3\,(m - 1) + (4 - \nu)$,

$$\begin{aligned} \Sigma \pm (f_1(a_1), f_2(a_1), f_3(a_1), b_1)\,(f_4(a_2), f_5(a_2), f_6(a_2), b_2) \ldots \\ (f_{3n-5}(a_{m-1}), f_{3m-4}(a_{m-1}), f_{3m-3}(a_{m-1})\, b_{m-1}) \\ \times \{f_{3m-2}(a_m), f_{3m-1}(a_m) \ldots f_{3n-\nu+1}(a_m), b_m\} = 0 \quad . \quad . \quad . \quad (560). \end{aligned}$$

For it is obviously no use retaining any term $(f_1(a_1), f_2(a_1), f_3(a_1), f_4(a_1))$, in which a b does not enter, as the minor array of this term has a column of zeros and vanishes.

We thus have three types of conditions for $\nu = 1$, 2 or 3, and these are of the forms, the functions F being linear,

$$\begin{aligned} &\text{I.} \quad (\mathrm{F}_1 a_m, \mathrm{F}_2 a_m, \mathrm{F}_3 a_m, b_m) = 0\,; \\ &\text{II.} \quad [\mathrm{F}_1 a_m, \mathrm{F}_2 a_m, b_m] = 0\,; \\ &\text{III.} \quad \{\mathrm{F}_1 a_m, b_m\} = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (561). \end{aligned}$$

In type I, if a_m is given, b_m lies in a plane; and a_m lies on a general cubic surface if b_m is given.

In type II, if a_m is given, b_m may be any point on a line; and if b_m is given, a_m may be any point on a twisted cubic.

In the third case, $F_1 a_m = t b_m$, and either point is determined if the other is given.

There is no difficulty in applying this method to the case of Art. 153. We must, however, include the case of four conditions being requisite. The last line must belong to a complex, a congruency, a ruled surface, or be one of a definite number of lines.

155. We shall now consider the critical cases when every first minor of (552) vanishes.

The minor obtained by omitting the last column expands into

$$\Sigma \pm (f_1(a_1), f_2(a_1), f_3(a_1), f_4(a_1)) \ldots \{f_{4m-3}(a_m), f_{4m-2}(a_m) \ldots f_n(a_m)\} \quad . \quad (562).$$

Here, as in the last article, we have the types

$$\begin{aligned} &\text{I.} \ (F_1 a_m, \ F_2 a_m, \ F_3 a_m, \ F_4 a_m) = 0\,; \\ &\text{II.} \ [F_1 a_m, \ F_2 a_m, \ F_3 a_m] = 0\,; \\ &\text{III.} \ \{F_1 a_m, \ F_2 a_m\} = 0\,; \\ &\text{IV.} \ F a_m = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (563) \end{aligned}$$

corresponding to $n = 4m$, $n = 4m + 1$, $n = 4m + 2$, and $n = 4m + 3$.

Now, from the nature of arrays, though it does not appear directly from the form of the expansion, these conditions are all combinatorial functions of the m points a.

I. In the first place, for the type I we have for $m = 1$ the Jacobian of a four-system. Next, for $n = 8$, $m = 2$ we have a one-conditioned assemblage of lines of the fourth order, or a complex of the fourth order. These are the lines which can be destroyed by single functions of the system. For $n = 12$, $m = 3$, (562) represents a one-conditioned assemblage of planes, and these planes envelope a surface of the fourth class, and each can be destroyed by a corresponding definite function of the system.

For $n = 16$, $m = 4$, the same equation represents a constant multiplied by the volume of the tetrahedron $(a_1 a_2 a_3 a_4)$ to the fourth power.

II. Again, for $n = 4m - 1$, and more particularly for $m = 1$, we have the critical sextic

$$[f_1 a f_2 a f_3 a] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (564),$$

of three functions; and for seven functions a congruency of lines common to a set of quartic complexes; while for eleven functions we have a two-conditioned assemblage of planes, or a developable of planes enveloping certain surfaces of the fourth class.

III. For $n = 4m - 2$ there is first the system of united points of $f_1^{-1} f_2$ for a pair of functions, or $\{f_1 a, f_2 a\} = 0$. Secondly, a ruled surface of lines destroyed by functions of a six-system; and thirdly, a determinate number of planes destroyed by

functions of a ten-system. For a fourteen-system it requires an invariant relation to vanish.

IV. This case requires a single function to destroy a point; it gives the lines destroyed by functions of a five-system (of these there are 20, compare Art. 114); and it imposes a condition on a nine-system of functions, so that some function of the system may be capable of destroying a plane. For a thirteen-system an invariant relation must vanish if a critical case arises for non-coplanar points.

I calculate the order of the Kummer surface of the quartic complex for the eight-system to be 72, and the order and class of the congruency of the double lines to be 24. The lines of this congruency would seem to be capable of being destroyed by two-systems of functions selected from the eight-system.

156. More particularly, if the line ab can be destroyed by a single function of an n-system,

$$\Sigma x_1 f_1 a = 0, \quad \Sigma x_1 f_1 b = 0 \quad \ldots \ldots \quad (565);$$

and the array

$$\begin{Bmatrix} f_1 a & f_2 a & \ldots f_n a \\ f_1 b & f_2 b & \ldots f_n b \end{Bmatrix} = 0 \quad \ldots \ldots \quad (566)$$

must vanish. The number of conditions is now $9 - n$, so that from a nine-system one function can be found to destroy an arbitrary line. For $n = 8$, we have the complex

$$\Sigma \pm (f_1 a f_2 a f_3 a f_4 a)(f_5 b f_6 b f_7 b f_8 b) = 0 \quad \ldots \ldots \quad (567).$$

If the plane a, b, c can be destroyed by a single function

$$\begin{Bmatrix} f_1 a & f_2 a \ldots f_n a \\ f_1 b & f_2 b \ldots f_n b \\ f_1 c & f_2 c \ldots f_n c \end{Bmatrix} = 0 \quad \ldots \ldots \quad (568),$$

and this requires $13 - n$ conditions. For $n = 12$ we have the surface enveloped by the plane (compare the last article)

$$\Sigma \pm (f_1 a f_2 a f_3 a f_4 a)(f_5 b f_6 b f_7 b f_8 b)(f_9 c f_{10} c f_{11} c f_{12} c) = 0 \quad . \quad . \quad (569).$$

157. When a line can be destroyed point by point by functions of a two-system selected from an n-system,

$$\Sigma (x_1 + t y_1) f_1 (a + tb) = 0, \text{ or } \Sigma x_1 f_1 a = 0, \ \Sigma x_1 f_1 b + \Sigma y_1 f_1 a = 0, \ \Sigma y_1 f_1 b = 0 \quad (570);$$

and the array

$$\begin{Bmatrix} f_1 a & f_2 a \ldots f_n a & 0 & 0 \ldots 0 \\ f_1 b & f_2 b \ldots f_n b & f_1 a & f_2 a \ldots f_n a \\ 0 & 0 \ldots 0 & f_1 b & f_2 b \ldots f_n b \end{Bmatrix} = 0 \quad \ldots \quad (571)$$

must vanish, or $13 - 2n$ conditions must be satisfied when the line is arbitrary. The

functions must satisfy $9 - 2n$ conditions, as the line may be made to satisfy four. For a four-system one condition must be satisfied for the existence of a line of this nature, but for a five-system (compare Art. 114) a ruled surface of such lines exists, triple chords of a curve of the tenth order.

If the line can be destroyed by functions of a three-system we have (compare Art. 114)

$$\Sigma (x_1 + ty_1 + t^2 z_1) f_1 (a + tb) = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad (572),$$

and the resulting array is of 4 rows and $3n$ columns, and vanishes if $13 - 3n$ conditions are satisfied. Finally, if the line is destroyed seriatim by functions of an included four-system, $21 - 4n$ conditions must be satisfied.

We may state that the number of conditions required to determine an N-system included in an n-system is

$$\mathrm{N}(n - \mathrm{N}) = \mathrm{N}'(n - \mathrm{N}'), \quad (\mathrm{N} + \mathrm{N}' = n) \quad . \quad . \quad . \quad . \quad . \quad (573).$$

158. As regards the destruction of planes, a plane may be destroyed *en bloc*, as in (568), or line by line, or point by point. In the second case,

$$\Sigma (x_1 + sy_1) f_1 (a + tb + sc + std) = 0,$$

$$\text{or} \quad \Sigma (x_1 + sy_1) f_1 (a + sc) = 0, \; \Sigma (x_1 + sy_1) f_1 (b + sd) = 0 \quad . \quad . \quad (574),$$

with the condition $(abcd) = 0$.

Thus the array is

$$\left\{ \begin{array}{ccccccc} f_1 a & f_2 a & .. & f_n a & 0 & ... & 0 \\ f_1 c & f_2 c & .. & f_n c & f_1 a & ... & f_n a \\ 0 & 0 & .. & 0 & f_1 c & ... & f_n c \\ f_1 b & f_2 b & .. & f_n b & 0 & ... & 0 \\ f_1 d & f_2 d & .. & f_n d & f_1 b & ... & f_n b \\ 0 & 0 & .. & 0 & f_1 d & ... & f_n d \end{array} \right\} = 0 \quad . \quad . \quad . \quad . \quad . \quad (575)$$

of 6 rows and $2n$ columns, requiring $25-2n$ conditions when we disregard $(abcd)=0$. This is the case in which *a function can destroy a hyperboloid* generator by generator.* The same number of conditions must be satisfied even when the four points are supposed co-planar.

Finally, the case in which the points are destroyed seriatim gives an array of $3n$ columns and 6 rows, requiring $25 - 3n$ conditions for its vanishing.

From these articles we can clearly trace the way in which a Jacobian of four functions may degrade, one of the most interesting being where it breaks up into a pair of quadrics, one of which is destroyed generator by generator by a two-system.

* In the paper on the interpretation of a quaternion as a point symbol, the equation $q = a + tb + sc + std$ is considered. It represents a ruled quadric and exhibits the dual generation.

SECTION XXI.

The Extension of the Method to Hyper-Space.

159. Exactly as in quaternions we may regard the sum of a scalar and a line vector in space of n dimensions as the symbol of a weighted point.

If

$$q = \mathrm{S}q + \mathrm{V}q = \left(1 + \frac{\mathrm{V}q}{\mathrm{S}q}\right)\mathrm{S}q = (1 + \mathrm{OQ})\,\mathrm{S}q\,;\ \mathrm{OQ} = \frac{\mathrm{V}q}{\mathrm{S}q} \quad . \quad . \quad (576);$$

q is the symbol of the point Q to which a weight $\mathrm{S}q$ is attributed.

The point represented by a sum of point symbols is the centre of mass of the weighted points, and the weight attributable to that point is the sum of the weights.

The equation

$$q = a + tb \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (577),$$

in which t is a variable scalar, is the equation of the line ab.

The most general homographic divisions on two lines ab and cd are represented by

$$q = a + tb, \quad q = c + td \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (578),$$

in which the weights $\mathrm{S}a$, $\mathrm{S}b$, $\mathrm{S}c$, $\mathrm{S}d$ have been suitably selected.

The equation

$$q = t_1a_1 + t_2a_2 + t_3a_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (579)$$

represents the plane of the points a_1, a_2, a_3; and more generally

$$q = t_1a_1 + t_2a_2 + \&c. \ldots + t_ma_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (580)$$

is the equation of the $(m - 1)$-flat containing the m points $a_1, a_2 \ldots a_m$.

I believe it is more convenient to call generally a plane space of m dimensions an m-flat, and to retain the name *plane* for its ordinary signification—a two-flat.

160. In accordance with Hamilton's notation ('Elements,' Art. 365) we propose to write

$$[a_1a_2a_3 \ldots a_m] = \mathrm{V}_m.\mathrm{V}a_1\mathrm{V}a_2 \ldots \mathrm{V}a_m - \Sigma \pm \mathrm{V}_{m-1}.\mathrm{V}a_2\mathrm{V}a_3 \ldots \mathrm{V}a_m\mathrm{S}a_1 \quad . \quad (581);$$

or briefly

$$[a]_m = \mathrm{V}_m[a]_n + \mathrm{V}_{m-1}[a]_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (582);$$

as the symbol of the $(m - 1)$-flat containing the m points $a_1, a_2 \ldots a_m$.

In order to justify this proposal, we observe that the array $[a_1a_2 \ldots a_m]$ changes sign whenever two contiguous elements are transposed. It consequently vanishes whenever one element is a scalar multiple of another, or whenever any group of elements is linearly connected by scalar coefficients; and it does not vanish under any other conditions. It is equivalent to the most general one-row array that can be formed from the m symbols a, because, according to the principles laid down on the subject of quaternion arrays, the general one-row array must be of the form

$$\{a_1a_2 \ldots a_m\} = x\mathrm{V}_m.\mathrm{V}a_1\mathrm{V}a_2 \ldots \mathrm{V}a_m + y\Sigma \pm \mathrm{V}_{m-1}.\mathrm{V}a_2\mathrm{V}a_3 \ldots \mathrm{V}a_m\mathrm{S}a_1 \quad . \quad (583);$$

and the separable parts V_m and V_{m-1} of $[a]_m$ afford all the information contained in the general array with indeterminate scalars x and y.

The equation of the flat containing m points a may be written in the form

$$[qa_1a_2a_3 \ldots a_m] = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (584),$$

as this implies (580)

$$q = t_1a_1 + t_2a_2 + t_3a_3 + \ldots + t_ma_m,$$

in which $t_1, t_2 \ldots t_m$ are variable scalars.

161. Returning to the relation (582)

$$[a]_m = \mathrm{V}_m[a]_m + \mathrm{V}_{m-1}[a]_m,$$

it is evident that $\mathrm{V}_{m-1}[a]_m$ is equal to the product of a scalar and a set of $m - 1$ mutually rectangular unit vectors $i_2, i_3 \ldots i_m$, in the $(m-1)$-flat containing the m points $a_1, a_2 \ldots a_m$. It is also apparent that $\mathrm{V}_m[a]_m$ is the product of a set of m mutually rectangular unit vectors in the m-flat containing the origin and the points a multiplied by a scalar. We may take this product of m vector units to be $i_1i_2i_3 \ldots i_m$. Thus we have

$$[a]_m = (yi_1 - x)\, i_2i_3 \ldots i_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (585),$$

where $i_si_t + i_ti_s = 0$, $i_s^2 = -1$, and where x and y are certain scalars. (Compare CLIFFORD'S 'Mathematical Papers,' p. 398.)

From this we find the symbol of a definite point

$$\mathrm{A}_m = 1 + \frac{\mathrm{V}_{m-1}[a]_m}{\mathrm{V}_m[a]_m} = 1 - \frac{xi_2i_3 \ldots i_m}{yi_1i_2i_3 \ldots i_m} = 1 - \frac{x}{yi_1} = 1 + \frac{x}{y}i_1 \quad . \quad (586);$$

and we verify at once that this point is a conjugate of all the points a with respect to the quadric

$$\mathrm{S}.q^2 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (587),$$

because for any one of these points we have

$$\mathrm{S}a_t\mathrm{A}_m = \mathrm{S}a_t\left(1 + \frac{\mathrm{V}_{m-1}[a]_m}{\mathrm{V}_m[a]_m}\right) = \mathrm{S}a_t + \frac{\mathrm{V}_ma_t\mathrm{V}_{m-1}[a]_m}{\mathrm{V}_m[a]_m} = \mathrm{S}a_t - \mathrm{S}a_t = 0 \quad . \quad (588),$$

as appears on reference to the equation (581).

In other words, A_m is the reciprocal in the m-flat which contains the origin and the points a of the $(m-1)$-flat which contains the points a.

For example, in three dimensions,

$$A_2 = 1 + \frac{Va_2Sa_1 - Va_1Sa_2}{VVa_1Va_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (589)$$

is the point in the plane oa_1a_2 which is reciprocal to the line a_1a_2.

162. A comparison of the equations (581) and (585) shows that the m points

$$x + yi_1, i_2, i_3 \ldots i_m \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (590)$$

(of which $i_2, i_3 \ldots i_m$ are at infinity) may be taken as defining the $(m-1)$-flat containing the points a.

Hence, conversely, if $[a]_m$ is any function satisfying the equations of condition

$$[a]_m = V_m[a]_m + V_{m-1}[a]_m; \quad \frac{V_{m-1}[a]_m}{V_m[a]_m} = V . \frac{V_{m-1}[a]_m}{V_m[a]_m} \quad . \quad . \quad . \quad (591),$$

it is the symbol of an $(m-1)$-flat. In fact, we can reduce this function to the form (585) and the proposition is evident by (590).

163. *The symbol of the flat reciprocal to* $[a]_m$ *with respect to the auxiliary quadric* (587), $S . q^2 = 0$, *in an n-space is*

$$[a]_m \Omega \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (592),$$

where Ω *is the product of "n" mutually rectangular vector units in the n-space*, *or*

$$\Omega = i_1i_2i_3 \ldots i_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (593).$$

In fact, from (585) we obtain

$$\begin{aligned}[a]_m \Omega &= (-)^{m-1}(yi_1 - x)\, i_1 i_{m+1} i_{m+2} \ldots i_n (i_2i_3i_4 \ldots i_m)^2 \\ &= (-)^{\frac{1}{2}m(m+1)}(y + xi_1)\, i_{m+1} i_{m+2} \ldots i_n = [a]_{n+1-m} \quad . \quad . \quad . \quad (594);\end{aligned}$$

and $n + 1 - m$ defining points of this new $(n - m)$-flat are (590)

$$y + xi_1, i_{m+1}, i_{m+2} \ldots i_n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (595).$$

But all these points are conjugates, with respect to the auxiliary quadric, of the m points (590); and therefore the flat $[a]_m\Omega$ is the reciprocal of the flat $[a]_m$.

More symbolically, we have the relations

$$V_m[a]_m . \Omega = V_{n-m} . [a]_m \Omega; \quad V_{m-1}[a]_m . \Omega = -V_{n-m+1} . [a]_m \Omega \quad . \quad (596);$$

and in particular for three dimensions we deduce the relations

$$[ab] = -(a'b'); \quad (ab) = [a'b'] \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (597),$$

connecting a line and its reciprocal (compare p. 224).

For odd spaces, if

$$n - m + 1 = m \quad \text{or} \quad m = \tfrac{1}{2}(n + 1),$$

the flat and its reciprocal, $[a]_m$ and $[a]_m\Omega$, are of the same order. This is the case for a line in three dimensions, and we recover from the general formulæ

$$[ab] = -(a'b'); \quad (ab) = [a'b'],$$

relations which I have elsewhere given connecting the symbols of reciprocal lines.

We are now prepared with all the necessary machinery for the geometry of flats and of their reciprocals.

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